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NUMERICAL APPROXIMATION FOR DIRECT SOLUTION OF THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS USING NEW CLASS OF FALKNER-TYPE BLOCK METHOD

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ABSTRACTS: *In this article, a new class of Falkner-type methods for direct solution of third order ordinary differential equation is developed. The approach of collocation and interpolation technique is adopted to derive the new Falkner-type methods, which is implemented in block mode to get approximation at grids points simultaneously. The resulting scheme is zero-stable, consistent and convergent with good region of absolute stability. The tabular and graphical presentations of the numerical results to the problems considered, demonstrate the effectiveness and good accuracy of the scheme in comparison with other methods*

KEYNOTE: Ordinary differential equation, third-order initial value problem, Falkner-type method

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INTRODUCTION

In this article, the direct method of solving third orders Initial Value Problem (IVPs) in Ordinary Differential Equations (ODEs) of the form.

$$\left.\begin{array}{c} y'''(x) = f(x, y(y), y'(x), y''(x), \qquad x \in [a, b]) \\ y(x_0) = \alpha, y'(x_0) = \beta, y''(x_0) = \gamma \end{array}\right\} \quad (1)$$

Where $y \in \mathbb{R}^N,\ f: \mathbb{R} * \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is continues vector functions, is considered.

Most of this kind of problem arises in physical sciences, engineering and social sciences which when formulated often lead to Ordinary Differential Equations (ODEs). Many of the model equation of problem of (1) may not be easily solve analytically, hence the need for numerical technique become a necessity. Earlier attempt have been made to solve the solution of (1) by method of reduction of order, but the method has some series of cost of implication and wastage of computer term. However, there are authors who make derived numerical technique for direct solution of problem (1) without reducing to the system of first order ODEs. [13], proposed a three-stage fifth-order Runge-Kutta method for direct solution of special third order ODEs with application to thin film flow problem. [12], Introduced new special two-derivative Runge-Kutta type methods involving the fourth derivatives of solving third order ODEs. [2], developed a new three-step method for numerical solution of third order ODEs via interpolation technique. [4], proposed a single step block method of P-stable for solving third order ODEs. [14] derived a four-step block method of hybrid linear multistep method for solution of third order ODEs. [10] proposed a numerical methods in block methods using the operational matrixes of Bernstein Polynomial for solution for the solution of third order ODEs. [13], Derived an improved sixth order multi-derivative block method using Legendre polynomial for solution of third order ODEs.

In this paper, the goal is to derive a new class of Falkner-type method for numerical approximation for direct solution of third order initial value problem without reducing to system of first order ODEs. Authors such as [7-9] and [11], proposed numerical method of Falkner –type method of second order ODEs but not of third order ODEs.

Derivation of the Method

In this section, three off step Falkner-type method for solution of (1) is presented below. Considering the trial solution given in the form:

$$Y(x) = \sum_{i=0}^{n} a_i P_i(x) \cong y(x) \qquad (2)$$

$$Y'(x) = \sum_{i=1}^{n} a_i P'_i(x) \qquad (3)$$

$$Y''(x) = \sum_{i=2}^{n} a_i P''_i(x) \qquad (4)$$

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$$Y'''(x) = \sum_{i=3}^{n} a_i P'''_i(x) \qquad (5)$$

Where n varies as the methods is developed and $P_i(x) = x^i$, is a power series, a_i is the coefficients to be determine.

Evaluating (2), (3), (4) at x_{v_1} and (3) at $i = 0, v_3, 1, v_2, 2, v_1, 3,4$ the system of equation is obtain as

$$\left.\begin{aligned} y_{n+v_1} &= Y(x_{n+v_1}) \\ y'_{n+v_1} &= Y'^{(x_{n+v_1})} \\ y''_{n+v_1} &= Y''^{(x_{n+v_1})} \\ y'''_{n+1} &= Y'''^{(x_{n+v_1})} = f(x_{n+v_1}) \end{aligned}\right\} \qquad (6)$$

Where, $v = k - r, v_1 = v - 1, v_2 = v_1 - 1, v_3 = v_2 - 1, i = 0, v_{3,}, 1, v_2, 2, v_1, 3, k.$ and k =step number, r =offstep point chosen.

The undetermined coefficients a_i are obtain by solving the system of equation (4) and the the obtained values are substitute into (2), yield the continues scheme

$$Y(x) = \alpha_i y_{n+i} + \alpha'_i h y_{n+i} + \alpha''_i \frac{h^2}{2!} y''_{n+i} + h^3 \left(\sum_{i=0}^{k} \beta_i(x) f_{n+k} + \beta_v(x) f_{n+v} \right) \qquad (7)$$

Similarly, substituting the coefficient into (3) yield the continues scheme

$$Y'(x) = \alpha_i y_{n+i} + \alpha'_i h y_{n+i} + h^2 \left(\sum_{i=0}^{k} \beta'_i(x) f_{n+k} + \beta'_v(x) f_{n+v} \right) \qquad (8)$$

Also, substituting the coefficient into (4) yield the continues scheme

$$Y''(x) = \alpha_i y_{n+i} + h \left(\sum_{i=0}^{k} \beta'_i(x) f_{n+k} + \beta'_v(x) f_{n+v} \right) \qquad (9)$$

Evaluating (7) – (9) at $x_n = x_{n+k}$, the main method is obtained. While the additional methods are also obtain by evaluation (7)-(9) at $x_{n+i} = 0, v, k$.

Main Method

The main formulas are obtained by substituting the values of $\alpha(x), \alpha'(x), a''(x)$ and β_i, β_v into (7) – (9) and evaluating $p(x_n + 4h)$, $p'(x_n + 4h)$ and $p''(x_n + 4h)$ to get approximation for $y(x_n + 4h), y'(x_n + 4h)$ and $y''(x_n + 4h)$. Thus, we obtain the following three off-step Falkner-type formulas

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$$\left.\begin{array}{l} y_{n+4} = y_{n+\frac{5}{2}} + \frac{3}{2}hy'_{n+\frac{5}{2}} + \frac{9}{8}h^2y''_{n+\frac{5}{2}} + h^3\left(\frac{19737}{2867200}f_n - \frac{9801}{179200}f_{n+\frac{1}{2}} + \frac{5427}{28672}f_{n+1} - \frac{33183}{89600}f_{n+\frac{3}{2}} + \frac{124983}{286720}f_{n+2} - \frac{22977}{179200}f_{n+\frac{3}{2}} + \frac{340659}{716800}f_{n+3} + \frac{24201}{2867200}f_{n+4}\right) \\ y'_{n+4} = y_{n+\frac{5}{2}} + \frac{3}{2}hy'_{\frac{5}{2}} + h^2\left(\frac{23643}{716800}f_n - \frac{2943}{11200}f_{n+\frac{1}{2}} + \frac{32757}{35840}f_{n+1} - \frac{20259}{11200}f_{n+\frac{3}{2}} + \frac{158301}{71680}f_{n+2} - \frac{501}{350}f_{n+\frac{5}{2}} + \frac{255033}{179200}f_{n+3} + \frac{35451}{716800}f_{n+4}\right) \\ y''_{n+4} = y''_{n+\frac{5}{2}} + h\left(\frac{23643}{716800}f_n - \frac{2943}{11200}f_{n+\frac{1}{2}} + \frac{32757}{35840}f_{n+1} - \frac{20259}{11200}f_{n+\frac{3}{2}} + \frac{158301}{71680}f_{n+2} - \frac{501}{350}f_{n+\frac{5}{2}} + \frac{255033}{179200}f_{n+3} + \frac{35451}{716800}f_{n+4}\right) \end{array}\right\} \quad (10)$$

These formulas are the main Falkner-type method for the solution of (1). For us to achieve that we need additional together with the main method to solve the required IVPs.

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Additional Methods.

Evaluating (7) at $x_{n+i}, i \coloneqq 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}.$ we obtain the formulas

$$\left.\begin{aligned}
y_n &= y_{n+\frac{5}{2}} - \frac{5}{2} h y'_{n+\frac{5}{2}} + \frac{25}{8} h^2 y''_{n+\frac{5}{2}} + h^3 \left(-\frac{3625}{1327104} f_n - \frac{8875}{193536} f_{n+\frac{1}{2}} - \frac{239375}{774144} f_{n+1} - \frac{122375}{290304} f_{n+\frac{3}{2}} - \frac{1954375}{1548288} f_{n+2} - \frac{113875}{193536} f_{n+\frac{5}{2}} + \frac{60875}{2322432} f_{n+3} + \frac{125}{344064} f_{n+4} \right) \\
y_{n+1} &= y_{n+\frac{5}{2}} - \frac{3}{2} h y'_{n+\frac{5}{2}} + \frac{9}{8} h^2 y''_{n+\frac{5}{2}} + h^3 \left(-\frac{1647}{2867200} f_n - \frac{891}{179200} f_{n+\frac{1}{2}} - \frac{423}{20480} f_{n+2} - \frac{1107}{89600} f_{n+\frac{3}{2}} - \frac{98577}{286720} f_{n+2} - \frac{35613}{179200} f_{n+\frac{5}{2}} + \frac{6291}{716800} f_{n+3} - \frac{351}{2867200} f_{n+4} \right) \\
y_{n+2} &= y_{n+\frac{5}{2}} - \frac{1}{2} h y'_{n+\frac{5}{2}} + \frac{1}{8} h^2 y''_{n+\frac{5}{2}} + h^3 \left(\frac{8959}{232243200} f_n + \frac{181}{537600} f_{n+\frac{1}{2}} + \frac{15689}{11612160} f_{n+1} - \frac{25201}{7257600} f_{n+\frac{3}{2}} - \frac{643}{7372873728} f_{n+2} - \frac{218881}{14515200} f_{n+\frac{5}{2}} + \frac{32267}{58060800} f_{n+3} + \frac{1807}{232243200} f_{n+4} \right) \\
y_{n+3} &= y_{n+\frac{5}{2}} - \frac{1}{2} h y'_{n+\frac{5}{2}} + \frac{1}{8} h^2 y''_{n+\frac{5}{2}} + h^3 \left(-\frac{15271}{232243200} f_n - \frac{383}{691200} f_{n+\frac{1}{2}} + \frac{541}{258048} f_{n+1} - \frac{34669}{7257600} f_{n+\frac{3}{2}} + \frac{60467}{7741440} f_{n+2} - \frac{110737}{483840} f_{n+\frac{5}{2}} + \frac{21629}{8294400} f_{n+3} + \frac{1501}{77414400} f_{n+4} \right) \\
y_{n+\frac{1}{2}} &= y_{n+\frac{5}{2}} - 2 h y'_{n+\frac{5}{2}} + 2 h^2 y''_{n+\frac{5}{2}} + h^3 \left(-\frac{61}{56700} f_n + \frac{13}{1575} f_{n+\frac{1}{2}} - \frac{55}{567} f_{n+1} - \frac{2308}{14175} f_{n+\frac{3}{2}} - \frac{51}{70} f_{n+2} - \frac{5221}{14175} f_{n+\frac{5}{2}} + \frac{233}{14175} f_{n+3} - \frac{13}{56700} f_{n+4} \right) \\
y_{n+\frac{3}{2}} &= y_{n+\frac{5}{2}} - h y'_{n+\frac{5}{2}} + \frac{1}{2} h^2 y''_{n+\frac{5}{2}} + h^3 \left(-\frac{461}{1814400} f_n + \frac{167}{75600} f_{n+\frac{1}{2}} + \frac{53}{6048} f_{n+1} - \frac{167}{8100} f_{n+\frac{3}{2}} - \frac{6193}{60480} f_{n+2} - \frac{2057}{25200} f_{n+\frac{5}{2}} + \frac{1633}{453600} f_{n+3} - \frac{31}{604800} f_{n+4} \right) \\
&\vdots
\end{aligned}\right\} \quad (11)$$

Similarly, evaluating (8) at $x_{n+i}, i \coloneqq 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}.$ we obtain the formulas

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$$
\left.
\begin{aligned}
y'_{n+\frac{1}{2}} &= y'_{n+\frac{5}{2}} - 2hy''_{n+\frac{5}{2}} + h^2\left(\frac{29}{56700}f_n + \frac{52}{4725}f_{n+\frac{1}{2}} + \frac{827}{2835}f_{n+1} + \frac{5708}{14175}f_{n+\frac{3}{2}} + \frac{349}{378}f_{n+2} + \frac{5494}{14175}f_{n+\frac{5}{2}} - \frac{241}{14175}f_{n+3} + \frac{13}{56700}f_{n+4}\right) \\
y'_{n+\frac{3}{2}} &= y'_{n+\frac{5}{2}} - hy''_{n+\frac{5}{2}} + h^2\left(\frac{59}{113400}f_n - \frac{167}{37800}f_{n+\frac{1}{2}} + \frac{187}{11340}f_{n+1} - \frac{167}{8100}f_{n+\frac{3}{2}} + \frac{314}{945}f_{n+2} + \frac{20851}{113400}f_{n+\frac{5}{2}} - \frac{467}{56700}f_{n+3} + \frac{13}{113400}f_{n+4}\right) \\
y'_{n} &= y'_{n+\frac{5}{2}} - \frac{5}{2}hy''_{n+\frac{5}{2}} + h^2\left(\frac{5725}{331776}f_n + \frac{3125}{12096}f_{n+\frac{1}{2}} + \frac{295625}{580608}f_{n+1} + \frac{24625}{36288}f_{n+\frac{3}{2}} + \frac{456875}{387072}f_{n+2} + \frac{9175}{18144}f_{n+\frac{5}{2}} - \frac{14375}{580608}f_{n+3} + \frac{125}{331776}f_{n+4}\right) \\
y'_{n+1} &= y'_{n+\frac{5}{2}} - \frac{3}{2}hy''_{n+\frac{5}{2}} + h^2\left(\frac{657}{16800}f_n - \frac{27}{3200}f_{n+\frac{1}{2}} + \frac{261}{5120}f_{n+1} + \frac{1017}{5600}f_{n+\frac{3}{2}} + \frac{44847}{71680}f_{n+2} + \frac{6429}{22400}f_{n+\frac{5}{2}} + \frac{2313}{179200}f_{n+3} + \frac{129}{716800}f_{n+4}\right) \\
y''_{n+2} &= y_{n+\frac{5}{2}} - \frac{1}{2}hy''_{n+\frac{5}{2}} + h^2\left(\begin{gathered}\frac{14639}{58060800}f_n - \frac{167}{75600}f_{n+\frac{1}{2}} + \frac{25913}{2903040}f_{n+1} - \frac{21157}{907200}f_{n+\frac{3}{2}} + \frac{18197}{276480}f_{n+2} + \frac{71599}{907200}f_{n+\frac{5}{2}} \\ - \frac{49891}{14515200}f_{n+3} + \frac{409}{8294400}f_{n+4}\end{gathered}\right) \\
y''_{n+3} &= y_{n+\frac{5}{2}} + \frac{1}{2}hy''_{n+\frac{5}{2}} + h^2\left(\begin{gathered}-\frac{27779}{58060800}f_n - \frac{2431}{604800}f_{n+\frac{1}{2}} + \frac{43937}{2903040}f_{n+1} - \frac{7753}{226800}f_{n+\frac{3}{2}} + \frac{21187}{387072}f_{n+2} - \frac{241807}{1814400}f_{n+\frac{5}{2}} \\ + \frac{49813}{2073600}f_{n+3} + \frac{8563}{58060800}f_{n+4}\end{gathered}\right)
\end{aligned}
\right\} \quad (12)
$$

Also, evaluating (9) at $x_{n+i}, i \coloneqq 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}.$ we obtain the formulas

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$$\left.\begin{aligned}
y''_{n+\frac{1}{2}} &= y''_{n+\frac{5}{2}} + h\left(\frac{4}{945}f_n - \frac{19}{105}f_{n+\frac{1}{2}} - \frac{68}{105}f_{n+1} - \frac{322}{945}f_{n+\frac{3}{2}} - \frac{68}{105}f_{n+2} - \frac{19}{105}f_{n+\frac{5}{2}} + \frac{4}{945}f_{n+3}\right) \\
y''_{n+\frac{3}{2}} &= y''_{n+\frac{5}{2}} + h\left(-\frac{1}{12096}f_n + \frac{109}{15120}f_{n+1} - \frac{367}{1890}f_{n+\frac{3}{2}} - \frac{701367}{1120}f_{n+2} - \frac{367}{1890}f_{n+\frac{5}{2}} + \frac{109}{15120}f_{n+3} + \frac{1}{12096}f_{n+4}\right) \\
y''_{n} &= y''_{n+\frac{5}{2}} + h\left(-\frac{57515}{387072}f_n - \frac{675}{896}f_{n+\frac{1}{2}} + \frac{15875}{96768}f_{n+1} - \frac{9925}{12096}f_{n+\frac{3}{2}} + \frac{2375}{7168}f_{n+2} - \frac{7564}{24192}f_{n+\frac{5}{2}} + \frac{3025}{96768}f_{n+3} - \frac{275}{387072}f_{n+4}\right) \\
y''_{n+1} &= y''_{n+\frac{5}{2}} + h\left(-\frac{117}{71680}f_n - \frac{81}{4480}f_{n+\frac{1}{2}} + \frac{4189}{17920}f_{n+1} - \frac{1179}{2240}f_{n+\frac{3}{2}} - \frac{19683}{35840}f_{n+2} - \frac{979}{4480}f_{n+\frac{5}{2}} + \frac{207}{17920}f_{n+3} - \frac{13}{71680}f_{n+4}\right) \\
y''_{n+2} &= y''_{n+\frac{5}{2}} + h\left(-\frac{1879}{1935360}f_n + \frac{23}{2688}f_{n+\frac{1}{2}} - \frac{1879}{53760}f_{n+1} + \frac{5671}{60480}f_{n+\frac{3}{2}} + \frac{37473}{107520}f_{n+2} - \frac{613}{2688}f_{n+\frac{5}{2}} + \frac{5909}{483840}f_{n+3} - \frac{13}{71680}f_{n+4}\right) \\
y''_{n} &= y''_{n+\frac{5}{2}} + h\left(-\frac{4183}{1935360}f_n + \frac{81}{4480}f_{n+\frac{1}{2}} - \frac{32719}{483840}f_{n+1} + \frac{9127}{60480}f_{n+\frac{3}{2}} - \frac{8419}{35840}f_{n+2} - \frac{55487}{120960}f_{n+\frac{5}{2}} + \frac{85973}{483840}f_{n+3} + \frac{275}{387072}f_{n+4}\right)
\end{aligned}\right\} \quad (13)$$

The main formulas and the additional formulas form the new class of three step hybrid Falkner-type methods in block form for the solution of third order differential equations.

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ANALYSIS OF THE METHODS

The linear differential operators associated with the proposed method is of the form

$$\mathcal{L}\{y(x):h\} \equiv Y(x) - \left[\alpha_i y_{n+i} + \alpha'_i h y_{n+i} + \alpha''_i \frac{h^2}{2!} y''_{n+i} + h^3 \left(\sum_{i=0}^{k} \beta_i(x) y'''_{n+k} + \beta_v(x) y'''_{n+v} \right) \right] \quad (14)$$

$y(x)$ is an arbitrary functions.

The Taylor series expansion of (14) around x yields

$$\left.\begin{aligned} \mathcal{L}(y(x)) &= \sum_{j=0}^{r+1} C_p h^p y^p + 0(h^{r+2}) \\ C_0 &= \sum_{j=0}^{k} \alpha_j \\ C_q(-1)^q &= \left[\frac{1}{q} \sum_{j=1}^{k} j\alpha_j + \frac{1}{(q-1)!} \sum_{j=0}^{k} j^{q-1} \beta_j \right] \\ q &= 1,2,3 \ldots \end{aligned}\right\} \quad (15)$$

Definition 1.

We say that the methods is of order $p \geq 1$, if $C_0 = C = \cdots C_p = C_{p+2} = 0$ and $C_{p+3} \neq 0$. In this case, expanding the proposed schemes in Taylor series yields the order of the method below

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$$\left.\begin{aligned}
C_{y_n} &= \frac{120125}{26159874048} y^{(11)}(x_n)h^{11} + 0(h^{12}) \\
C_{y_{n+\frac{1}{2}}} &= \frac{31}{10644480} y^{(11)}(x_n)h^{11} + 0(h^{12}) \\
C_{y_{n+1}} &= \frac{12501}{807403520} y^{(11)}(x_n)h^{11} + 0(h^{12}) \\
C_{y_{n+\frac{3}{2}}} &= \frac{3403}{5109350400} y^{(11)}(x_n)h^{11} + 0(h^{12}) \\
C_{y_{n+2}} &= \frac{4387}{43599790080} y^{(11)}(x_n)h^{11} + 0(h^{12}) \\
C_{y_{n+3}} &= -\frac{2868693011}{653996851200} y^{(11)}(x_n)h^{11} + 0(h^{12}) \\
C_{y_{n+4}} &= -\frac{223371}{8074035200} y^{(11)}(x_n)h^{11} + 0(h^{12})
\end{aligned}\right\} \quad (16)$$

The order of the methods have order $p = 7$ (See [10])

Zero-stability and convergence

This is the concept concerning the behavior of a numerical method with stability of the first characteristic polynomial as $h \to 0$. To analyze the zero-stability of the proposed method, the roots of the first characteristics polynomial as $h \to 0$ must be simple or less than 1.

The proposed scheme can be writtenin matrix form as

$$A^0\tilde{Y}_\mu - A'\tilde{Y}_{\mu-1} = 0$$

Where

$$\bar{Y}_\mu = (y_{n+1}, y_{n+2}, \dots, y_{n+k})^T$$

$$\bar{Y}_{\mu-1} = (y_n, y_{n+1}, y_{n+k-r})^T$$

A^0 is the identity matrix. Following the procedure in Ramose, (2019) the proposed methods can be shown that

$$\boldsymbol{A^0} = \begin{bmatrix} 1000000 \\ 0100000 \\ 0010000 \\ 0001000 \\ 0000100 \\ 0000010 \\ 0000001 \end{bmatrix}, A^1 = \begin{bmatrix} 0000100 \\ 0000100 \\ 0000100 \\ 0000100 \\ 0000100 \\ 0000100 \\ 0000100 \end{bmatrix}$$

$$\rho(r) = |rA^0 - A'|$$

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$$\rho(r) = \left[r\begin{bmatrix} 1000000 \\ 0100000 \\ 0010000 \\ 0001000 \\ 0000100 \\ 0000010 \\ 0000001 \end{bmatrix} - \begin{bmatrix} 0000100 \\ 0000100 \\ 0000100 \\ 0000100 \\ 0000100 \\ 0000100 \\ 0000100 \end{bmatrix} \right]$$

$$\rho(r) = \begin{bmatrix} r & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & r & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & r & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & r & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & r-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & r & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & r \end{bmatrix}$$

$$\rho(r) = (r-1)r^6, r = 0,0,0,0,0,0,1$$

Thus, the proposed methods is zero-stable

Theorem 1. A Linear Multistep Methods (LMMs) is said to be convergent if it is consistent and zero-stable [10]

Remarks: The roots of the proposed schemed is obtained as $\rho(r) = r^6(r-1), r = 0000001$. This implies that. The proposed methods is convergent.

Region of Absolute Stability

As we mentioned before, zero-stability is a concept concerning the behavior of a numerical method for $h \to 0$. In order to know if a numerical method will give reasonable result for a given $h > 0$, we need a concept of stability different from zero-stability. Considering the stability function inform

$$M(z) = \mathfrak{h}(A - Cz - Dz^2 - Ez^3) - B$$

where $z = \lambda h$ and $\mathrm{A, B, C, D, E}$ are obtained from interpolating and collocating points of the method. Computing the stability functions and its first derivative give the polynomial which can be plotted via Matlab environment.

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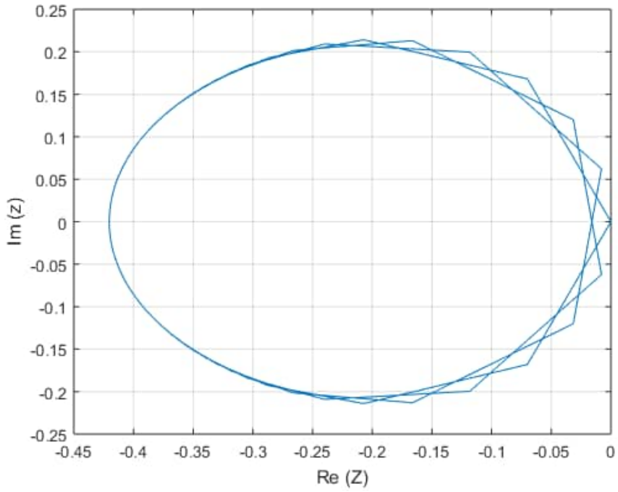

Figure 1. Region of Absolute Stability

Implementation of the Method

In this section, the proposed method will be applied to solve third order initial value problems. With the notation below, we presents numerical and graphical obtained for some test problems.

FBM: Error in Falkner-type Block Methods

EISBM (2020): Error in [5]

Problem 1 This nonlinear third order value problem

$$\left.\begin{array}{c} y''' - xy'' + x^2y^2 = xsinx - cosx + x^2cos^2x \\ y(0) = 0, y'(0) = 1, y''(0) = 0, h = 0.1 \\ Exact\ solution\ y(x) = \sin x \end{array}\right\}$$

Table 1. Numerical and Error Result for Problem 1.

x	Exact solution	Numerical results	FBM	[1]
0.1	0.099833416646828152	0.099833416646828087	6.5E-17	6.661338E-16
0.2	0.19866933079506122	0.19866933079506091	3.1E-16	3.913536E-15
0.3	0.29552020666133958	0.29552020666133885	7.3E-16	1.243450E-14
0.4	0.38941834230865049	0.38941834230864887	1.62E-15	2.886580E-14
0.5	0.47942553860420300	0.4794255386041972700	5.73E-15	5.601075E-14
0.6	0.56464247339503536	0.5646424733950211572	1.420E-14	9.692247E-13
0.7	0.64421768723769105	0.6442176872376637780	2.272E-14	1.546541E-13
0.8	0.71735609089952276	0.7173560908994772668	4.549E-14	2.325917E-13
0.9	0.78332690962748339	0.7833269096274118517	7.145E-14	3.346212E-13
1.0	0.84147098480789651	0.8414709848077896323	1.0688E-14	4.644063E-13

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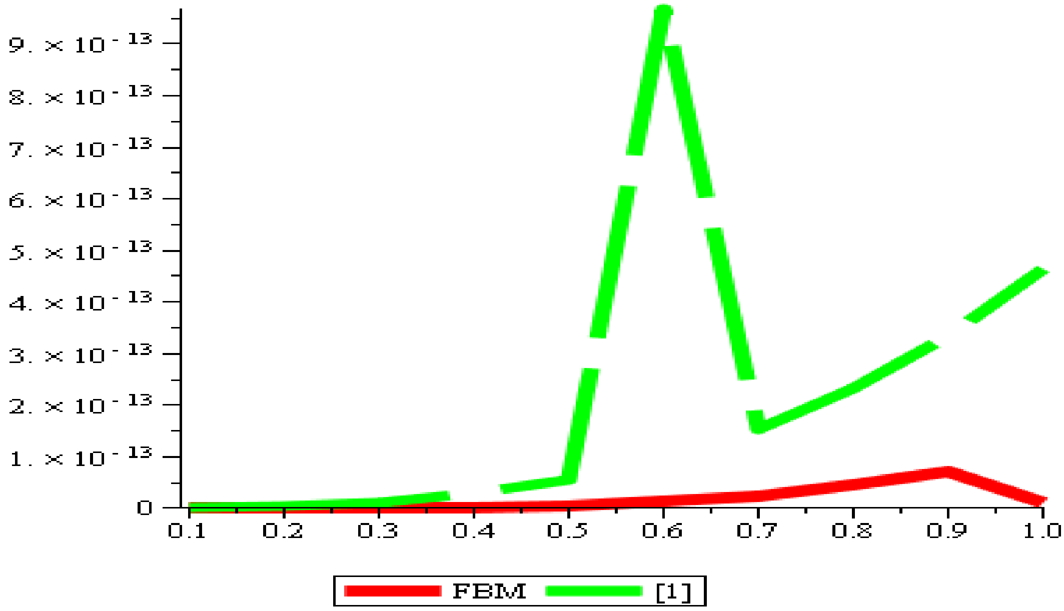

Fig. 2: Showing Error Results of Problem 1.

Problem 2

$$y''' - 3\sin(x) = 0, \quad y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.1$$

Exact solution $y(x) = 3\cos(x) + \left(\frac{x^2}{2}\right) - 2.$

Table 2. Numerical and Error Result for Problem 2.

x	Exact Solution	Numerical Results	FBM	[2]	EIBSBM [5]
0.1	0.9900124958340773	0.99001249583407733	3.000E-17	1.5834E-09	5.9885430E-13
0.2	0.9601997335237249	0.96019973352372504	1.400E-16	1.9544E-08	3.8212766E-12
0.3	0.9110094673768181	0.91100946737681841	3.100E-16	9.0423E-08	9.5831121E-12
0.4	0.8431829820086552	0.84318298200865607	8.700E-16	2.7267E-07	1.7947976E-11
0.5	0.7577476856711182	0.75774768567112142	3.220E-15	6.4622E07	3.26226124E-11
0.6	0.6560068447290349	0.65600684472904331	8.410E-15	1.3115E06	6.0369598E-11
0.7	0.5395265618534653	0.53952656185348154	1.624E-14	2.3884E-06	1.0098744E-11
0.8	0.4101201280414963	0.41012012804152356	2.726E-14	4.0154E-06	1.5461399E-10
0.9	0.2698299048119934	0.26982990481203964	4.624E-14	6.349E-06	2.2891933E-10
1.0	0.1209069176044192	0.12090691760449411	7.494E-14	9.5554E-06	3.3474887E-10

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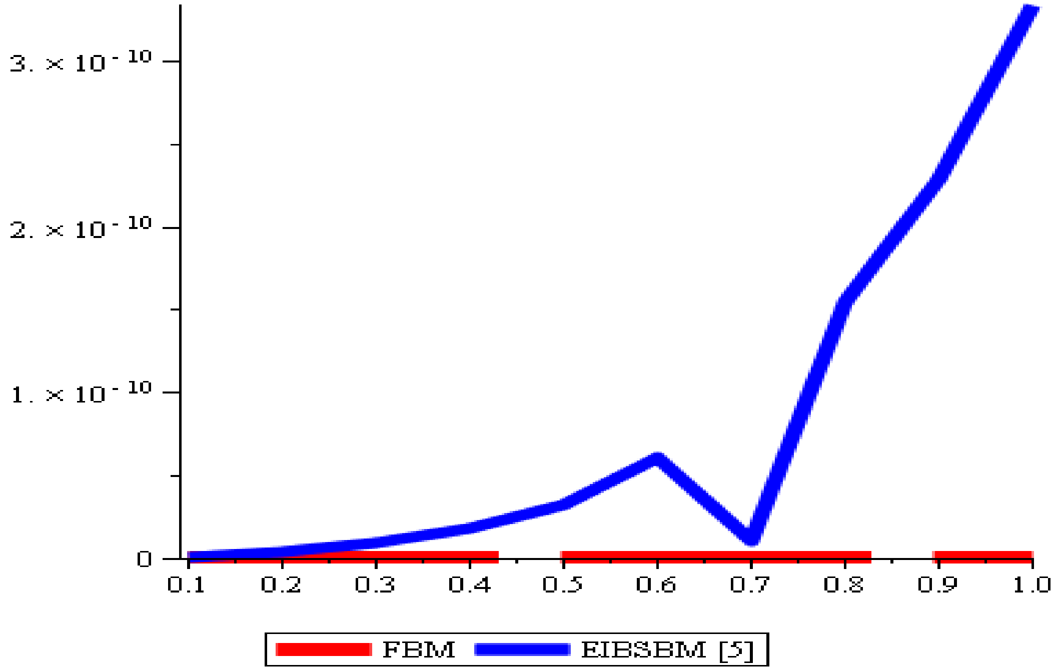

Fig. 3: Showing Error Results of Problem 2

Problem 3

$$y''' + 4y' - x = 0, \qquad y(0) = 0, y'(0) = 0, y''(0) = 1, h = 0.1$$

$$y(x) = -\frac{3}{16}\cos(2x) + \frac{3}{16} + \frac{x^2}{8}$$

Table 3. Numerical and Error Result for Problem 3.

x	Exact Solution	Numerical Solution	FBM	[2]
0.1	0.0049875166547671900	0.0049875166547591617	8.028300E-15	4.8615E-09
0.2	0.019801063624459050	0.019801063624421787	3.725300E-14	6.8948E-08
0.3	0.043999572204435320	0.043999572204348743	8.657700E-14	3.3046E-07
0.4	0.076867491997208534	0.076867491997208534	1.979460E-13	1.0011E-06
0.5	0.11744331764972380	0.11744331764893174	7.920600E-13	2.3493E-06
0.6	0.16455792103562370	0.16455792103360603	2.017670E-12	4.6774E-06
0.7	0.21688116070620482	0.21688116070237873	3.826090E-12	8.2949E-06
0.8	0.27297491043149164	0.27297491042524128	6.250360E-12	1.3491E-05
0.9	0.33135039275495382	0.33135039274477906	1.017476E-11	2.0506E-05
1.0	0.39052753185258920	0.39052753183678863	1.580057E-11	2.9502E-05

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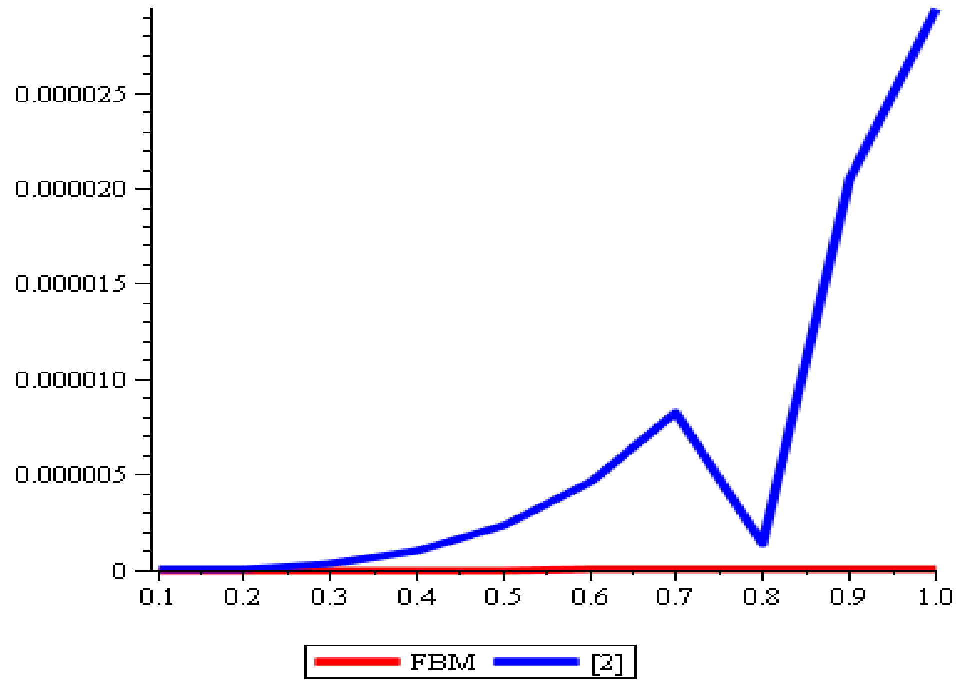

Fig.4: Showing Error Results of Problem 3

Problem 4

$$y''' - y'' + y' - y = 0, y(0) = 1, y'(0) = 0, y''(0) = -1, h = 0.01$$

$$Exacts\ Solutiom: y(0) = cosx$$

Table 4. Numerical and Error Result for Problem 4.

x	Exact Solution	Numerical Solution	**FBM**	[2]	EIBSBM [5]
0.01	0.999950000416665278	0.99995000041666528	2.00E-17	7.1067E-13	0.00000000+00
0.02	0.999800006666577778	0.99980000666657778	2.00E-18	4.7071E-12	5.5511151E-16
0.03	0.999550033748987516	0.99955003374898752	4.00E-18	1.3033E-11	8.7707619E-15
0.04	0.999200106660977940	0.99920010666097794	0.000+00	2.7412E-11	6.4614980E-14
0.05	0.998750260394966247	0.99875026039496625	3.00E-18	5.4703E-11	2.6290081E-14
0.06	0.998200539935204166	0.99820053993520417	4.00E-18	9.7557E-11	-
0.07	0.997551000253279575	0.99755100025327957	-5.00E-18	1.5823E-10	-
0.08	0.996801706302619385	0.99680170630261938	-5.00E-18	2.4843E-10	-
0.09	0.995952733011994253	0.99595273301199425	-3.00E-18	3.7250E-10	-
0.10	0.995004165278025766	0.99500416527802576	-6.00E-18	5.3320E-10	-

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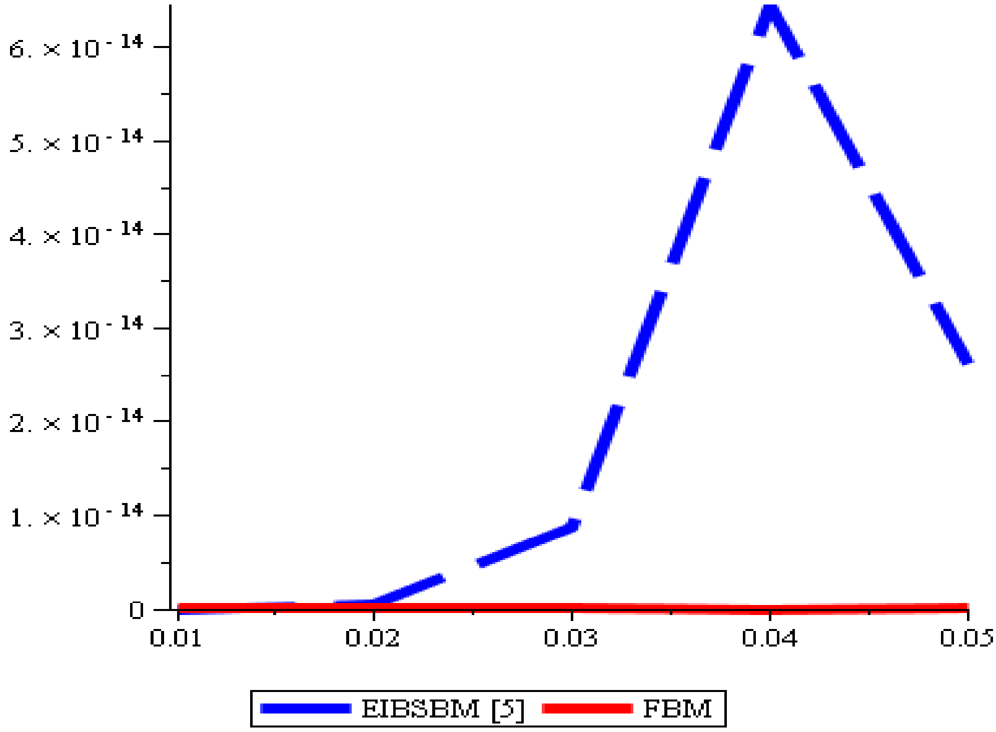

Fig 5: Showing Error Results of Problem 4

DISCUSSION AND CONCLUSION

From the analysis of the derived schemed above, shows that the new methods is of order six (6), Zero-stable, Consistent and has good region of absolute stability. The numerical and error results as observed from the tables above, shows that the derived numerical methods least error.

CONCLUSION

In this study, Numerical Approximation for Direct Solution of Third-Order Ordinary Differential Equations Using New Class of Falkner-Type Block Method has been proposed. The methods effectively solve third order differential equations. The tabular and graphical presentation of our results considered demonstrates the effectiveness of the method.

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