RELATIVISTIC MECHANICS IN GRAVITATIONAL FIELD WITHIN OBLATE SPHEROIDAL COORDINATES BASED UPON RIEMANNIAN GEOMETRY FOR ROTATING HOMOGENEOUS MASS DISTRIBUTION

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ABSTRACT: *The emergence of the geometrical theory of gravitation (general relativity) by Albert Einstein in his quest to unite special relativity and the Newtonian law of universal gravitation has led to several Mathematical approaches for the exact and analytical solution for all gravitational fields in nature. The first and the most famous analytical solution was the Schwarzchild's which can be constructed by finding a mapping where the metric tensor takes a simple form i.e. the vanishing of the non-diagonal elements. In this paper, we construct exact solution of the Einstein geometrical gravitational field equation using Riemannian metric tensor called the golden metric tensor that was first developed by Howusu, in the year 2009, for the rotating homogeneous mass distribution within oblate Spheroidal Coordinates. The equations of motion for test particles in the Oblate Spheroidal Geometry were derived using the coefficient of affine connection. Then the law of conservation of momentum and energy are equivalently formulated using the generalized Lagrangian as compared to the analytical solution of the Schwarzchild's gravitational field. We also derived the planetary equation of motion in the equatorial plane of the Oblate Spheroidal body for this gravitational field.*

KEYWORD: Relativistic mechanics, Gravitational Field, Riemannian geometry, Oblate Spheroidal Coordinates.

INTRODUCTION

The geometrical theory of gravitation called the General Relativity was first published by Albert Einstein in 1915 [1, 2, 3]. This theory unifies Special Relativity and Sir Isaac Newton's law of universal gravitation with the assumption that gravitation is not due to a force but due to the manifestation of curved space-time, energy-mass and momentum content of the space time. Immediately after Einstein's geometrical field equation was published in 1915, the search for their exact and analytical solution for all the gravitational fields in nature began [4].

The first approach to the development of the exact analytical solutions of Einstein's geometrical gravitational field equations was explored such that the metric tensor assumed a simple form, so that the off-diagonal elements vanished. This method led to the first analytical solution, called the famous Schwarzchild solution [4]. The second method was developed by Weyl and Levi-Cevita, based on the assumption that the metric tensor contains symmetries of assumed forms of the associated killing vectors [4, 5, 6, 7, 8, 9, 10, 11]. Then the third method was to set up a Taylor series expansion of some initial value hyper surface, based on the consistent initial value data. This method was void because it cannot generate a successful solution [4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

Howusu and Chifu in the year 2000 were able to introduce a new method and approach to the construction of the exact analytical solution of Einstein's geometrical gravitational field equations as an extension of the Schwardchild analytical solution. Schwarzchhild's metric is well-known to be a metric tensor due to a static spherically symmetric body given as [14]:

$$g_{00} = 1 + \frac{2f(r)}{c^2} \tag{1}$$

$$g_{11} = -\left[1 + \frac{2f(r)}{c^2}\right]^{-1} \tag{2}$$

$$g_{22} = -\, r^2 \tag{3}$$

$$g_{33} = -\, r^2 \sin^2 \Theta \tag{4}$$

and

$$g_{\mu V} = 0;\ \text{Otherwise} \tag{5}$$

where

$$f(r) = \frac{2GM}{r} \tag{6}$$

and r > R, the radius of the space spherical mass, G is the universal gravitational constant, M is the total mass of the distribution and c is the speed of light in vacuum. It may be noted that f is a function of the radial coordinate (r) only; because the distribution and hence its exterior gravitational field possess spherical symmetry. It may also be noted that these metric components should be reduced to the field of a point mass located at the origin and contain or having Newton's equations motion in the gravitational field of the spherical body; It follows that, f (r) is approximately equal to the Newtonian gravitational scalar potential in the extensor region of the body, $\Phi\ (r)$ [14].

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Chifu et al. [15] in their article determined the general relativistic mechanics in gravitational fields produced by homogenous mass distribution rotating with constant angular velocity about a fixed diameter within a static spherical body placed in an empty space. The metric tensor used for this gravitational field is given as (15);

$$g_{00} = 1 + \frac{2f(r,\theta)}{c^2} \quad (7)$$

$$g_{11} = -\left[1 + \frac{2f(r\theta)}{c^2}\right]^{-1} \quad (8)$$

$$g_{22} = -r^2 \quad (9)$$

$$g_{33} = -r^2 \sin^2\theta \quad (10)$$

$$g_{\mu V} = 0; \text{otherwise} \quad (11)$$

where f (r, θ) is an arbitrary function determined by the mass distribution within the spherical body. The solution of the field equations obtained by this metric tensor gives explicit expressions for the function of f (r, θ). In this paper, we are out to study the general relativistic mechanics in gravitational fields produced by homogenous rotating mass distribution for an Oblate Spheroidal body placed in an empty space, using the Golden Metric tensor based upon Riemannian geometry.

METHODS

Coefficient of affine connection

Consider a static homogeneous Spheroidal body of total mass or of M and density, ρ. Here, we assume the mass or pressure distribution within the Spheroidal is homogenous and rotating with uniform velocity about a fixed diameter. For this mass distribution, it is important to note that our arbitrary function f will be independent of the coordinate of time and azimuthal angle. Thus, the coordinates for metric tensor in Oblate Spheroidal coordinate for this gravitational is given as [16, 17]:

$$g_{00} = -\left[1 + \frac{2}{c^2}f(\xi,\eta)\right] \quad (12)$$

$$g_{11} = \frac{a^2(\eta^2+\xi^2)}{1-\eta^2}\left[1 + \frac{2}{c^2}f(\xi,\eta)\right]^{-1} \quad (13)$$

$$g_{22} = \frac{a^2(\eta^2+\xi^2)}{1-\xi^2}\left[1 + \frac{2}{c^2}f(\xi,\eta)\right]^{-1} \quad (14)$$

$$g_{33} = a^2(1-\eta^2)(1+\xi^2)\left[1 + \frac{2}{c^2}f(\xi,\eta)\right]^{-1} \quad (15)$$

$$g_{NV} = 0; \text{otherwise} \quad (16)$$

where $f(\xi,\eta)$ is an arbitrary function determined by mass distribution within the Oblate Spheroidal. It may be noted that the golden metric tensor satisfies Einstein's field equation and the invariance of the line element; by virtue of their construction [11,12].

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The contravariant golden metric tensor for this gravitational field can be obtained by using the Quotient Theorem of Tensor Analysis as [18]:

$$g_{00} = -\left[1 + \frac{2}{c^2} f(\eta, \xi)\right]^{-1} \tag{17}$$

$$g_{11} = \left[\frac{a^2(\eta^2+\xi^2)}{1-\eta^2}\right]^{-1} \left(1 + \frac{2}{c^2} f(\eta, \xi)\right) \tag{18}$$

$$g_{22} = \left[\frac{a^2(\eta^2+\xi^2)}{1-\xi^2}\right]^{-1} \left(1 + \frac{2}{c^2} f(\eta, \xi)\right) \tag{19}$$

$$g_{33} = [a\,(1-\eta^2)1+\xi^2]^{-1} \left(1 + \frac{2}{c^2} f(\eta, \xi)\right) \tag{20}$$

$$g_{\mu V} = 0; \text{otherwise} \tag{21}$$

Here, we shall use the coefficient of affine connection for all gravitational field in nature which are expressed in term of the metric tensor as [19, 20]:

$$\Gamma^{\propto}_{\mu v} = \frac{1}{2} g^{\propto \in} [g_{\mu\epsilon,v} g_{\in v,\mu} - g_{\mu v,\in}] \tag{22}$$

where the comma denotes partial differentiation with respect to x^{α}, x^{μ} *and* x^{ν}. Hence, we compute explicitly expressions for the coefficients of affine connection based upon the golden metric tensor in this gravitational fields as:

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{C^2}\left[1 + \frac{2}{C^2} f(\eta, \xi)\right] f(\eta, \xi), 1 \tag{23}$$

$$\Gamma^0_{02} = \Gamma^0_{20} = \frac{1}{C^2}\left[1 + \frac{2}{C^2} f(\eta, \xi)\right] f(\eta, \xi), 2 \tag{24}$$

$$\Gamma^{11}_{00} = -\frac{1}{C^2}\left[\frac{a^2(\eta^2+\xi^2)}{1-\eta^2}\right]^{-1} \left(1 + \frac{2}{C^2} f(\eta, \xi)\right) f,(\eta, \xi), 1 \tag{25}$$

$$\Gamma^{11}_{11} = \frac{\xi}{\eta^2+\xi^2} + \frac{1}{C^2}\left[1 + \frac{2}{C^2} f(\eta, \xi)\right]^{-1} f(\eta, \xi), 1 \tag{26}$$

$$\Gamma^{11}_{21} = \frac{\xi}{\eta^2+\xi^2} + \frac{1}{C^2}\left[1 + \frac{2}{C^2} f(\eta, \xi)\right]^{-1} f(\eta, \xi), 2 \tag{27}$$

$$\Gamma^{11}_{33} = \frac{\eta\,(1+\xi^2)(1-\eta^2)}{a\,(\eta^2+\xi^2)} + \frac{1}{C^2}\,\frac{(1-\eta^2)^2\,(1+\xi^2)}{a\,(\eta^2+\xi^2)}\left[1 + \frac{2}{C^2} f(\eta, \xi)\right]^{-1} f(\eta, \xi), 1 \tag{28}$$

$$\Gamma^{12}_{00} = \frac{1}{C^2}\left[\frac{a^2(\eta^2+\xi^2)}{1-\eta^2}\right]^{-1} \left(1 + \frac{2}{C^2} f(\eta, \xi)\right) f,(\eta, \xi), 2 \tag{29}$$

$$\Gamma^{12}_{11} = \left[\frac{1-\eta^2}{a^2(\eta^2+\xi^2)}\right]^{-1} \left(1 + \frac{2}{C^2} f(\eta, \xi)\right) f,(\eta, \xi), 2 \tag{30}$$

$$\Gamma^{12}_{12} = \Gamma^{12}_{21} = \frac{\xi(1-\eta^2)}{\eta^2+\xi^2(1+\xi^2)} - \frac{1}{C^2}\left[1 + \frac{2}{C^2} f(\eta, \xi)\right]^{-1} f(\eta, \xi), 1 \tag{31}$$

$$\Gamma^{12}_{22} = \frac{\eta}{\eta^2+\xi^2} + \frac{1}{C^2}\left[1 + \frac{2}{C^2} f(\eta, \xi)\right]^{-1} f(\eta, \xi), 2 \tag{32}$$

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$$\Gamma_{13}^{13} = \Gamma_{31}^{13} = \frac{\xi}{1+\xi^2} - \frac{1}{C^2}\left[1 + \frac{2}{C^2} f(\xi,\eta)\right]^{-1} f,1 \quad (33)$$

$$\Gamma_{23}^{13} = \Gamma_{32}^{13} = \frac{\eta}{1+\xi^2} - \frac{1}{C^2}\left[1 + \frac{2}{C^2} f(\xi,\eta)\right]^{-1} f(\xi,\eta)2 \quad (34)$$

$$\Gamma_{N\upsilon}^{\propto} = 0;\ \text{otherwise} \quad (35)$$

Thus, the gravitational field exterior to a homogeneous region of Oblate Spheroidal geometry based upon the golden metric tensors has twelve distinct non-zero coefficients of affine connection. These coefficients of affine connection are very useful in the construction of the Riemannian equation of motion for particles of non-zero rest mass.

Motion of Test Particles, Based upon the Riemannian Geometry

A test mass can be defined as a body that is so small in which the gravitational field produced by it is so negligible that it does not have any effect on the space metric. It may also be viewed as a continuous body, which is approximated by its geometrical center and has nothing in common with a point mass whose density should obviously be infinite [21].

According to the Theory of General Relativity, equation of motion for particles of non-zero rest masses as given by [18, 19, 22] as:

$$\frac{d^2x^{\propto}}{d\tau^2} + \Gamma_{N\upsilon}^{\propto}\left(\frac{dx^N}{d\tau}\right)\left(\frac{dx^{\upsilon}}{d\tau}\right) = 0 \quad (36)$$

where τ is the proper time. To obtain the equations of motion for the test particles, we take the following steps: setting $\propto = 0$ in (36) and substituting (27) and (23) give the time equations of motion as:

$$\ddot{t} + \frac{2}{C^2}\left[1 + \frac{2}{C^2} f(\xi,\eta)\right] \dot{t}\dot{\xi} f(\xi\eta)1$$

$$+\frac{2}{C^2}\left[1 + \frac{2}{C^2} f(\xi,\eta)\right] \dot{t}\dot{\eta} f(\xi\eta)2 = 0 \quad (37)$$

where the dot denotes differentiation with respect to proper time. For motion confined in the equatorial plane of the body $\eta = 0$, similar to Schwarzchild's time equation when $f(r,\theta)$ reduced to $f(r)$. The third term of (37) can only contribute to the rotation of the mass within the Spheroidal geometry. Hence, the time equation (37) is reduced to:

$$\ddot{t} + \frac{2}{C^2}\left[1 + \frac{2}{C^2} f(\xi,\eta)\right] \dot{t}\dot{\xi} f(\xi,\eta) = 0 \quad (38)$$

It may be noted that (37) can be written equivalently as:

$$\frac{d}{\tau}(ln\dot{t}) + \frac{d}{d\tau}\left[ln\left(1 + \frac{2}{C^2} f(\xi,\eta)\right)\right] = 0 \quad (39)$$

By integration, (39) becomes:

$$\dot{t} = B\left[1 + \frac{2}{C^2} f(\xi,\eta)\right]^{-1} \quad (40)$$

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where B is the constant of integration but as $t \rightarrow \xi,\ f(\xi,\eta) \rightarrow 0$ and thus the constant B is equivalent to unity. This is the expression for the variation of the time on a clock moving in the gravitational field. It is also the same as that obtainable in Schwarzchild's gravitational field [18, 19, 22].

Similarly, putting $\propto = 1$ in (33), we obtain the radial equation of motion as:

$$\ddot{\xi} + \frac{1}{C^2}\left[\frac{a^2(\eta^2+\xi_2)}{1-\eta^2}\right]^{-1}\left(1+\frac{2}{C^2}f\right) f_{,1}\,\dot{t}^2 + \left[\frac{\xi}{(\eta^2+\xi_2)} + \frac{1}{C^2}\left(1+\frac{2}{C^2}f\right)^{-1} f_{,1}\right]\dot{\xi}^2$$
$$+\left[\frac{\xi}{(\eta^2+\xi_2)} - \frac{1}{C^2}\left(1+\frac{2}{C^2}f\right)^{-1} f_{,2}\right]\dot{\eta}\dot{\xi} + \left[\frac{\eta(1+\xi^2)(1-\eta^2)}{a(\eta^2+\xi_2)} + \frac{1}{C^2}\frac{(1-\xi^2)(1+\eta^2)}{a(\eta^2+\xi_2)}\left(1+\frac{2}{C^2}f\right)^{-1} f_{,1}\right]\dot{\phi}^2$$
(41)

For pure radial motion $\dot{\eta} = \dot{\phi} = 0$ and hence (41) reduces to:

$$\ddot{\xi} + \left[\frac{\xi}{(\eta^2+\xi_2)} - \frac{1}{C^2}\left(1+\frac{2}{C^2}f\right)^{-1} f_{,}\right]\dot{\xi}^2 = 0 \quad (42)$$

Equation (42) can be used to obtain the instantaneous speed of the particle of non-zero rest mass in this gravitational field.

Also, putting $\propto = 2$, (33) become:

$$\ddot{\eta} + \frac{1}{C^2}\left[\frac{a^2(\eta^2+\xi_2)}{1-\eta^2}\right]^{-1}\left(1+\frac{2}{C^2}f(\eta,\xi)\right) f(\eta,\xi)_{,2}\,(\dot{t})^2$$
$$+\left[\frac{\eta(1+\xi^2)}{(\eta^2+\xi^2)(1+\eta^2)} - \frac{1}{C^2}\left(\frac{1-\xi^2}{1+\eta^2}\right)\left(1+\frac{2}{C^2}f(\eta,\xi)\right)^{-1} f(\eta,\xi)_{,2}\right]\dot{\xi}^2$$
$$+\left[\frac{\xi(1-\xi^2)}{(\eta^2+\xi^2)(1+\xi^2)} - \frac{1}{C^2}\left(\frac{1-\xi^2}{1+\eta^2}\right)\left(1+\frac{2}{C^2}f(\eta,\xi)\right)^{-1} f\right] f_{,1}\,\dot{\xi}\dot{\eta}$$
$$+\left[\frac{\eta}{(\eta^2+\xi_2)} - \frac{1}{C^2}\left(1+\frac{2}{C^2}f(\eta,\xi)\right) f(\eta,\xi)_{,2}\right]\dot{\eta}^2$$
(42)

Lastly, putting $\propto = 3$, (33) becomes:

$$\ddot{\phi} + \left[\frac{\xi}{1+\xi^2} - \frac{1}{C^2}\left(1+\frac{2}{C^2}f(\eta,\xi)\right)^{-2} f_{,1}\right] \qquad \dot{\xi}\dot{\phi} + \left[\frac{-\eta}{1+\xi^2} - \frac{1}{C^2}\left(1+\frac{2}{C^2}f(\eta,\xi)\right)^{-2} f_{,2}\right]\dot{\eta}\dot{\phi}$$
(44)

It may be noted that (43) and (44) are the equation of motion for the particles of non-zero rest masses along the η and ϕ axis.

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Orbits

It follows from the theory of General Relativities, the Lagrangian in the space of exterior to any mass distribution within Oblate Spheroidal geometry in space time is defined by [22]:

$$L = \frac{1}{c}\left[-g_{N\upsilon}\left(\frac{dx^{N}}{\partial\tau}\right)\left(\frac{dx^{\upsilon}}{\partial\tau}\right)\right]^{\frac{1}{2}} = 0 \tag{45}$$

where τ is the proper time. therefore follows that;

$$L = \frac{1}{c}\left[-g_{00}\, C^{2}\left(\frac{\partial t}{\partial\tau}\right)^{2} - g_{11}\left(\frac{\partial\xi}{\partial\tau}\right)^{2} - g_{22}\left(\frac{\partial\eta}{\partial\tau}\right)^{2} - g_{33}\left(\frac{\partial\phi}{\partial\tau}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} = 0$$
(46)

Since $\eta = 0$ along the equatorial plane, then (46) becomes:

$$L = \frac{1}{x}\left[-g_{00}\, C^{2}\left(\frac{\partial t}{\partial\tau}\right)^{2} - g_{11}\left(\frac{\partial\xi}{\partial\tau}\right)^{2} - g_{33}\left(\frac{\partial\phi}{\partial\tau}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{47}$$

By subtracting the covariant the metric tensor, in the equatorial of the homogeneous Oblate Spheroidal body, it follows that:

$$L = \frac{1}{c}\left[\left(1 + \frac{2}{c^{2}}f\ (\xi\eta)(\dot{t})\right)\right]^{\frac{1}{2}} + \frac{1}{c^{2}}\left(1 + \frac{2}{c^{2}}f\ (\xi,\eta)\right)^{-1}\dot{\xi}^{2} - a^{2}(1+\xi^{2})\left(1 + \frac{2}{c^{2}}f\ (\xi,\eta)\right)^{-1}\dot{\phi}^{2}$$
(48)

where the dot donates differentiation with respect to proper time.

Since the gravitational field is conservative, it follows that the Euler-Lagrange equations of motion for a conservative system in which the potential energy is independent of the generalized velocities is written as [22]:

$$\frac{\partial L}{\partial \varkappa^{\propto}} = \frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial \varkappa^{\propto}}\right) \tag{49}$$

But,

$$\frac{\partial L}{\partial \varkappa^{0}} \equiv \frac{\partial L}{c\partial t} = 0 \tag{50}$$

It can be shown according to the time homogeneity of the field that;

$$\frac{\partial L}{\partial \dot{t}} = \text{constant} \tag{51}$$

Then, it follows from (51) that (47) becomes:

$$\left[1 + \frac{2}{c^{2}}f\ (\xi,\eta)\right]\ \dot{t} = k \tag{52}$$

Hence

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$$\dot{k} = 0 = \left[1 + \frac{2}{c^2} f\ (\xi, \eta)\right]\ \ddot{t} \tag{53}$$

where k is of a constant.

It is instructive to note that the Lagrangian for this gravitational field is invariant to azimuthal and hence angular rotation because space is isotropic in nature and hence angular momentum is conserved, thus:

$$\frac{\partial L}{\partial \dot{\phi}} = 0 \tag{54}$$

It follows that:

$$\frac{\partial L}{\partial \dot{\phi}} = a^2\ (1 + \xi^2)\ \left(1 + \frac{2}{c^2} f\ (\xi, \eta)\right)^{-1}\ \dot{\phi} = l \tag{55}$$

where l is a constant called the angular momentum in the equatorial plane of the gravitational fields. Then:

$$\dot{l} = 0 = \frac{a^2}{c}(1 + \xi^2)\ \left(1 + \frac{2}{c^2} f\ (\xi, \eta)\right)^{-1}\ \ddot{\phi} \tag{56}$$

This is the law of the conservation of angular momentum in the equatorial plane of an oblate plane based upon the golden metric tensors.

It is well known that the orbits in Schwarzchild's space-time, the Lagrangian for permanent orbits in the equatorial plane [7] is given as:

$$L = \left[\left(1 - \frac{2GM}{C^2 r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{C^2}\left[\left(1 - \frac{2GM}{C^2 r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2\right] + r^2 \frac{d\phi^2}{d\tau}\right]^{\frac{1}{2}}$$
(57)

For Time-like orbits, the result of the Lagrangian for the planetary equation of motion in Schwarzschild's space Time is given as:

$$\frac{d^2 u}{d\phi^2} + u = \frac{GM}{\hbar^2} + 3\frac{GM}{C^2} u^2 \tag{58}$$

where $u = \frac{1}{r}\ and\ h$ is a constant of motion. The solution of (58) depicts the famous perihelion precession of planetary orbits [2, 16, 18] for a given time-like orbits.

Likewise, for rotating homogeneous distribution of masses within the spherical body, the planetary equation of motion in the equation plane of this gravitational field is given by [13]:

$$\frac{d^2 u}{d\phi^2} - 2u\ (1 + u^2)^{-1}\ \frac{du}{d\phi} + u\ (1 + u^2)^2\ \left(1 + \frac{2}{c^2} f\ (u, \theta)\right) = \left(\frac{2t^2}{l^2} - \frac{u^2}{l^2}\right)(1 + u^2)^2\ \frac{\partial f}{\partial u}$$
(59)

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For Time like orbits, (59) becomes:

$$\frac{d^2u}{d\phi^2} - 2u\,(1+u^2)^{-1}\,\frac{du}{d\phi} + u\,(1+u^2)^2\,\left(1+\frac{2}{c^2}f\,(u,\theta)\right) = \left(\frac{2}{l^2} - \frac{u^2}{l^2}\right)(1+u^2)^2\,\frac{\partial f}{\partial u}$$
(60)

where $u = \frac{1}{r(\phi)}$ and it is well known (13), that the Lagrangian L = ϵ, with $\epsilon = 1$ for Time-Like orbits and $\epsilon = 0$ for null or bits.

Now setting L= ϵ in (48) and squaring both sides, and then, hence the Lagrangian in the equatorial plane in the gravitational field exterior to a rotating mass distribution within Oblate Spheroidal geometry is given as:

$$\epsilon^2 = \frac{1}{C^2}\left[\left(1+\frac{2}{c^2}f\,(\xi,\eta)(\dot{t})^2\right)\right]$$

$$+\frac{1}{c^2}\left[-\,a^2\,\xi^2\,\left(1+\frac{2}{c^2}f\,(\xi,\eta)\right)^{-1}\right]\dot{\xi}^2 - a^2(1+\xi^2)\left(1+\frac{2}{c^2}\,f\,(\xi,\eta)\right)^{-1}\dot{\phi}^2$$
(61)

Substituting (52) and (55) into (61) and simplifying to yield:

$$\xi^2 + \frac{l^2\left(1+\frac{2}{c^2}f\,(\xi,\eta)\right)^{-2}}{a^{\psi}\,c^2\,\xi^2(1+\xi^2)} = \frac{k^2}{a^2\,c^2\,\xi^2}\,\frac{\epsilon^2}{a^2\,\,\xi^2}\,\left(1+\frac{2}{c^2}\,f\,(\xi,\eta)\right) \quad (62)$$

In general relativity, we are more interested in the shape of orbits i.e as a function of the azimuthal angle [2, 15, 18]. Hence, we transform (62) into an equation in terms of the azimuthal angle, ϕ. Now, let:

Then: $\xi = r(\phi)$ (63)

And

$$\mathrm{U}(\phi) = \frac{1}{\xi\,(\phi)} \quad (64)$$

$$\dot{\xi} = \frac{1}{1+u^2}\,\frac{du}{d\phi} \quad (65)$$

By imposing that the transformations of (64) and (65) on (62) and simplifying, we have:

$$\left(\frac{1}{1+u^2}\right)^2\,\left(\frac{du}{d\phi}\right)^2 + \frac{l^2\,u^{\psi}\left(1+\frac{2}{c^2}f\,(\xi,\eta)\right)^2}{a^{\psi}\,c^2(1+u^2)} = \frac{k^2\,u^2}{a^2\,c^2} - \frac{\epsilon^2\,u^2}{a^2}\,\left(1+\frac{2}{c^2}\,f\,(u,\eta)\right)$$
(66)

Equation (66) should have been integrated immediately but it leads to elliptical integrals, which are awkward to handle [16]. We thus differentiate (66) to obtain:

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$$\frac{d^2u}{d\phi^2} - 2u\,(1+u^2)^{-1}\,\frac{du}{d\phi} + \frac{u^3\,(2+u^2)}{a^{\Psi}\,c^2}\left(1+\frac{2}{c^2}\,f\,(u,\eta)\right)^2$$

$$= \frac{u\,(1+u^2)^2}{a^2}\left[\frac{k^2}{c^2} - \epsilon^2\right] - \frac{u^3}{c^2}\,(1+u^2)\,\left[2\left(1+\frac{2}{c^2}\,f\,(u,\eta)\right) + \frac{\epsilon^2(1+u^2)}{a^{\Psi}\,c^2}\right]\frac{\partial}{\partial u}\,f\,(u,\eta)$$
(67)

For time-like orbits, (65) reduces to:

$$\frac{d^2u}{d\phi^2} - 2u\,(1+u^2)^{-1}\,\frac{du}{d\phi} + \frac{u^3\,(2+u^2)}{a^{\Psi}\,c^2}\left(1+\frac{2}{c^2}\,f\,(u,\eta)\right)^2$$

$$= \frac{u\,(1+u^2)^2}{a^2}\left[\frac{k^2}{c^2} - 1\right] - \frac{u^2}{c^2}\,(1+u^2)\,\left[2\left(1+\frac{2}{c^2}\,f\,(u,\eta)\right) + \frac{(1+u^2)}{a^{\Psi}\,c^2}\right]\frac{\partial}{\partial u}\,f\,(u,\eta)$$
(67)

This is the planetary equation of motion in the equatorial plane of the oblate Spheroidal body in this gravitational field. For null orbits, since light travel on null geodesics and we choose $\epsilon = 0$, (67) reduces to:

$$\frac{d^2u}{d\phi^2} - 2u\,(1+u^2)^{-1}\,\frac{du}{d\phi} + \frac{u^3\,(2+u^2)}{a^{\Psi}\,c^2}\left(1+\frac{2}{c^2}\,f\,(u,\eta)\right)^2$$

$$= \frac{u\,(1+u^2)^2k^2}{a^2\,c^2} - \frac{2\,u^2}{c^2}\,(1+u^2)\left(1+\frac{2}{c^2}\,f\,(u,\eta)\right)\frac{\partial}{\partial u}\,f\,(u,\eta)$$
(68)

As the photon equation of motion in the vicinity of the homogenous rotating mass distribution within a static oblate spheroidal body is based upon Riemannian geometry using the Golding Metric tensor. In the limit of $c,^0$ (66) reduces to the well-known Euclidean geometry for oblate spheroidal coordinate as:

$$\frac{d^2u}{d\phi^2} - 2u\,(1+u^2)^{-1}\,\frac{du}{d\phi} + = 0 \qquad (69)$$

The general solution of (68) is taken to be a perturbation of the solution of (69). Hence, the immediate result of this analysis is that it will produce an expression for the total deflection of light grazing the massive Spheroidal.

CONCLUSION

Equation (33), (37), and (40) are the equations of motion for test particles in the gravitational field exterior to a homogeneous rotating mass distribution within a static Oblate Spheroidal.

We also obtained the expression for the conservation of energy and angular momentum as (57) and (54) respectively. Then, the planetary equations of motion and the photon equations of motion in the vicinity of the mass were also obtained as equations (66) and (67).

The immediate theoretical, physical and astrophysical implications of the results obtained in this paper are as follow:

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- The results obtained in (66) and (68) extend the Schwarzschild's and Chifu's et al. solutions from the well-known gravitational fields of pure spherical bodies to those of spheroidal bodies and hence the Spheroidal effect. These equations are therefore open up for further research work and astrophysical interpretation.

- Finally, the work in this paper is an elegant demonstration of deriving the planetary equations of motion in the equatorial plane for an orthogonal curvilinear coordinate system other than the well-known Cartesian, cylindrical and spherical body.

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