Advanced Journal of Science, Technology and Engineering
Volume 4, Issue 1, 2024 (pp. 52-66)
www.abjournals.org

SOLUTIONS TO FOURIER TRANSFORMS METHOD TO COMPLEX VARIABLE OF NON-HOMOGENEOUS FRACTIONAL DIFFERENTIAL EQUATIONS

Kalu Uchenna

Mathematics/College of Physical and Applied Sciences (COLPAS), Michael Okpara University of Agriculture, Umudike.

Email: uchennakalu304@gmail.com

Cite this article:

Kalu Uchenna (2024), Solution to Fourier Transforms Method to Complex Variable of Non-homogeneous Fractional Differential Equations. Advanced Journal of Science, Technology and Engineering 4(1), 52-66. DOI: 10.52589/AJSTE-3EVF5SPX

Manuscript History

Received: 23 Jan 2024

Accepted: 8 Mar 2024

Published: 3 Apr 2024

ABSTRACT: *This work is devoted to the study of fractional differential equations involving Caputo non-homogeneous fractional differential equations. Using Fourier transform method, a complex variable explicit solution to non-homogeneous second-order fractional differential equation was obtained.*

KEYWORDS: Fractional differential equation, Mittag-Leffler function, Riemann-Liouville fractional equations, Caputo derivative, Fourier transforms.

Article DOI: 10.52589/AJSTE-3EVF5SPX
DOI URL: https://doi.org/10.52589/AJSTE-3EVF5SPX

INTRODUCTION

Fractional calculus is a branch of mathematics investigating the properties of derivatives and integrals of non-integer orders called fractional derivatives and integrals. The history of fractional calculus was first mentioned in Leibniz's letter to L'Hospital in the year 1695, which related to his generalisation of meaning of the notation $\left(d^n / dx^n\right) d$ for the derivative of order $n \in N,\ N = 0,1,2,...,$ when $n = 1/2$? , means he was interested to the derivative of order $1/2$, where the idea of semi-derivative was suggested. During that time, fractional calculus was built on formal foundations by many famous Mathematicians, like Liouville, Grunwald, Riemann, Euler, Lagrange, Heaviside, Fourier, etc. many of them proposed original approaches, which can be found chronologically [1]. Many works have been done on fractional calculus in the derivation of particular solutions of a significantly large number of linear and non-linear ordinary and partial differential equations. The fractional integral may be used for describing the cumulation of some quantity, when the order of integration is unknown, it can be determined as a parameter of a regression model as presented [2]. One of the major reasons for fractional calculus is that it can be considered as a super set of integer-order calculus of the second and higher order. Other applications occur in the following fields: fluid flow, viscoelasticity, diffusive transport akin to diffusion, probability, statistics and electrical networks dynamical processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing as well as control theory of dynamical systems, among others [3]. Many physical systems appear to exhibit fractional order behaviour that may vary with time or space. The fractional calculus has generated the operations of differentiation and integration to any fractional order. The order may take on any real or imaginary value. Some applications of fractional calculus amount to replacing the time derivative in a given evolution equation by a derivative of fractional order. Interesting attempts have been made recently to give the physical meaning to the initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives proposed. Thus, fractional calculus has the potential to accomplish what integer-order calculus cannot. It has been believed that many of the great future developments will come from the applications of fractional calculus to differential fields. Some recent benefits to the theory of fractional differential equations can be seen in [3-6].

The first attempt was probably to introduce a fractional Fourier transform started by Wiener in the paper [7] published as early as 1929. The main contribution of [7] was in a discussion of a relation between an expansion of a function in a series of orthogonal Hermite polynomials and its Fourier transformation whereas the introduced fractional Fourier transform was just a byproduct of the used method. From the mathematical point of view, the fractional Fourier transform in [3, 6, 7] and in many other publications was based on the fractional powers of the Fourire transform defined through the set of its eigenfunctions given by the Gauss-Hermite functions. Moreover, the fractional Fourier transform of this type permits also the following interpretation for its applications in applied mathematics and physics, and especially in filter design, signal analysis and pattern recognition. For the theory of the Fourier transform on the space of tempered distributions and on the space $L_p(\mathbb{R}),\ 1 \le p \le 2$ we refer the reader to [8] and [9] respectively.

Our aim is thus to apply the Fourier transform method to construct nonhomogeneous fractional differential equations. We consider the fractional Caputo-type derivative of this form

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$$\left({}^{c}D_{0+}^{\alpha}q\right)(t)+\lambda q(t)=h(t) \tag{1.1}$$

Where, $h(t)$ is a $n-times$ continuously differentiable function and ${}^{c}D_{0+}^{\alpha}q$ is the fractional Caputo-type derivative of order $\alpha\in(n-1,n),\ n\in N^{+}$

Preliminaries

We introduce some basic definitions, notations of fractional integral calculus.

Definition 2.1. (The Riemann-Liouville fractional integral of order $\alpha>0$). This fractional integral of order $\alpha>0$ for a function $u(t)\in C^{1}\left([0,b],\mathbb{R}^{n}\right);b>0$ is given by

$${}^{RL}D_{0}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\eta)^{\alpha-1}u(\eta)d\eta,\ \ 0<\alpha<\infty \tag{2.1.1}$$

Where, $\Gamma(.)$ denote the Euler's Gamma function. The left and right Riemann-Liouville derivative with order $\alpha>0$ of a given function $u(t)\in C^{1}\left([0,b],\mathbb{R}^{n}\right)$ are respectively given as

$${}^{RL}D_{0,t}^{\alpha}u(t)=\frac{d^{n}}{dt^{n}}\left[D_{a,t}^{\alpha-n}u(t)\right]=\frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dt^{n}}\int_{a}^{t}(t-\eta)^{n-\alpha-1}u(\eta)d\eta \tag{2.1.2}$$

and

$${}^{RL}D_{t,b}^{\alpha}u(t)=(-1)^{n}\frac{d^{n}}{dt^{n}}\left[D_{t,b}^{\alpha-n}u(t)\right]=\frac{(-1)^{n}}{\Gamma(n-\alpha)}\frac{d^{n}}{dt^{n}}\int_{t}^{b}(\eta-t)^{n-\alpha-1}u(\eta)d\eta \tag{2.1.3}$$

Where n is an integer which satisfies $n-1\leq\alpha<n$

Definition 2.2. (The Caputo derivative of fractional order $\alpha>0$). The Caputo derivative of fractional order $\alpha>0$ is defined as

$${}^{c}D_{a+}^{\alpha}h(t)=\int_{a}^{t}\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}h(s)ds\ ,\ n-1<\alpha<n\ ,\ n=[a]+1 \tag{2.2.1}$$

where $[a]$ denote the integer part of the real number α .

Definition 2.3. (Caputo differential operator ${}^{c}D^{a}$). Caputo differential operator ${}^{c}D^{a}$ is given by

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$$ {}^{c}D^{a}z(t) = {}^{RL}D^{a}\sum\left[z - T_{[a]-1}z\right](t) \tag{2.3.1}$$

Where,

${}^{RL}D^{a}z(t) = \frac{d^{[\alpha]}}{dt[\alpha]}\frac{1}{\Gamma([\alpha]-\alpha)}\int_0^t (t-s)^{[\alpha]-\alpha-1}z(s)\,ds$ is the Riemann-Liouville differential operator and $T_n z(t) = \sum_{k=0}^{n}\frac{z^{(k(\ ))}(o)}{k!}t^k$ is the nth degree Taylor polynomial for z, centred at the origin. Here, ${}^{c}D^{a}$ is tacitly assumed to be a left-sided differential operator and to have its standing point at $t=0$, so that one naturally seek solution to the differential equation on an interval of the form $[0,T]$ with some $T>0$

Definition 2.4. (see [14]). The one dimensional fractional Fourier transform with rotational angle of function $q(t)\in L^1(\mathbb{R})$ is given by

$$\Im_\alpha[q(t)](\omega) = \hat{q}_\alpha(\omega) = \int_{\mathbb{R}} K_\alpha(t,\omega)q(t)\,dt,\ \omega\in\mathbb{R} \tag{2.4.1}$$

Where, the kernel

$$K_\alpha(t,\omega) = \begin{cases} C_\alpha \exp\left(i\left(q^2+\omega^2\right)\cot\alpha)/2\right) - iq\omega\cos ec\alpha, & if\quad \alpha \neq n\pi \\ \frac{1}{\sqrt{2\pi}}\exp(-iq\omega), & if\quad \alpha = \frac{\pi}{2} \end{cases} \tag{2.4.2}$$

$$C_\alpha = \sqrt{\frac{1-i\cot\alpha}{2\pi}} \tag{2.4.3}$$

The inversion formula of (2.4.1) is given by

$$q(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{K}_\alpha(t,\omega)\hat{q}_\alpha(\omega)\,d\omega,\ \ \mathrm{t}\in\mathbb{R} \tag{2.4.4}$$

Where, the kernel

$$\hat{K}_\alpha(t,\omega) = \begin{cases} C'_\alpha \exp\left(-i\left(q^2+\omega^2\right)\cot\alpha)/2\right) + iq\omega\cos ec\alpha, & if\quad \alpha \neq n\pi \\ \frac{1}{\sqrt{2\pi}}\exp(iq\omega), & if\quad \alpha = \frac{\pi}{2} \end{cases} \tag{2.4.5}$$

$$C'_\alpha = \sqrt{2\pi(1+i\cot\alpha)} \tag{2.4.6}$$

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Definition 2.5. (Error Function). The Error can be defined as

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt, \quad x \in \mathbb{R} \tag{2.5.1}$$

Definition 2.6. (The Mittag-Leffler function). We defined the Mittag-Leffler function as

$$E_{\alpha,\beta}(z) = \sum \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (Z, \alpha, \beta \in \mathbb{C},\ \Re(\alpha) > 0) \tag{2.6.1}$$

Definition 2.7. (The Wright function). The Wright function is defined by

$$\phi(\alpha, \beta; z) = \sum \frac{1}{\Gamma(\alpha k + \beta)} \cdot \frac{z^k}{k!}, \qquad (z, \alpha, \beta \in \mathbb{C}) \tag{2.7.1}$$

Definition 2.7. (The Binomial Coefficients). The Binomial Coefficients are defined by

$$\binom{\lambda}{n} = \frac{\lambda!}{\lambda!(\lambda - n)!} = \frac{\lambda(\lambda-1)(\lambda-n+1)}{n!} \tag{2.7.1}$$

Where, λ and n are integers. Observe that $0! = 1$, then $\binom{\lambda}{0} = 1,\ \binom{\lambda}{\lambda} = \frac{\lambda!}{\lambda!(\lambda-\lambda)!} = 1$, also $(1-z)^{-\lambda} = \sum_{r=0}^{\infty} \frac{(\lambda)r}{r!} z^r = \sum_{r=0}^{\infty} \binom{\lambda + r - 1}{r} z^r$

Definition 2.8. (The Gamma Function). The basic interpretation of the Gamma function is simply the generalisation of the factorial for all real numbers. It is defined by

$$\Gamma(x) = \int_0^{\infty} \exp(-t) t^{x-1} dt, \quad x \in \mathbb{R}^+ \tag{2.8.1}$$

Definition 2.9. (The Beta function). The Beta function can be defined in terms of Gamma function as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y \in x \in \mathbb{R}^+ \tag{2.9.1}$$

It can also be defined in term of a definite integral as

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y \in x \in \mathbb{R}^+ \tag{2.9.2}$$

Definition 2.10. (The Fourier transform $\Im$ and its inverse transform $\Im^{-1}$). The Fourier transform $\Im$ and it inverse transform $\Im^{-1}$ of $f(t),\ t \in (-\infty, \infty)$ is defined by

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$$\Im[f](w) = \hat{f}(w) = \int_{-\infty}^{+\infty} \exp(i\omega t) f(t) dt \qquad \omega \in \square \tag{2.10.1}$$

and

$$\Im^{-1}[f](t) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-i\omega t) \hat{f}(\omega) d\omega \qquad t \in \square \tag{2.10.2}$$

Definition 2.11. (Convolution Theorem). The theorem states that the Laplace transform of the convolution of two functions is the product of their Laplace transforms. If $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$ respectively, then

$$f * g = L\left\{\int_0^t f(t-z) g(z) dz = F(s) G(s)\right\} \tag{2.11.1}$$

Definition 2.12. (The fractional integral). The fractional integral of $y(t)$ of order α is defined as

$$D^{-\alpha} y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-z)^{\alpha-1} y(z) dz, \quad \alpha > 0 \tag{2.12.1}$$

Equation (2.12.1) is actually a convolution integral. So using (2.12.1), we find that

$$L\left\{D^{-\alpha} y(t)\right\} = \frac{1}{\Gamma(\alpha)} L\left\{t^{\alpha-1}\right\} L\left\{y(t)\right\} = S^{-\alpha} Y(s). \quad \alpha > 0 \tag{2.12.2}$$

Equation (2.12.2) is the Laplace transform of the fractional integral. We see for $\alpha > 0,\ \mu > -1$ that

$$L\left\{D^{-\alpha} t^{\mu}\right\} = \frac{\Gamma(\mu+1)}{S^{\mu+\alpha+1}} \text{ and } L\left\{D^{-\alpha} e^{\beta t}\right\} = \frac{1}{S^{\alpha}(S-\beta)} \tag{2.12.3}$$

Definition 2.13. (The fractional Fourier transform $\hat{f}_{\alpha}(\omega)$). The fractional Fourier transform $\hat{f}_{\alpha}(\omega)$ of order $\alpha > 0$ is defined as

$$[\Im f_{\alpha}](\omega) = \hat{f}_{\alpha}(\omega) = \int_{-\infty}^{+\infty} f(t) e_{\alpha} xp(\omega, t) dt \tag{2.13.1}$$

Where,

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$$e_\alpha xp(\omega,t)=\exp\left(isign(\omega)\right)|\omega|^{1/\alpha}x=\begin{cases}\exp\left(-i|\omega|^{1/\alpha}x\right), & \omega\le 0\\ \exp\left(i|\omega|^{1/\alpha}x\right), & \omega\ge 0\end{cases} \tag{2.13.2}$$

MAIN RESULT

Proposition 3.4. Let v be a real nonnegative number and let f be piecewise continuous on $j'=(0,\infty)$ and integrable on any finite subinterval of $j'=[0,\infty]$. Then for $t,v>0$, we defined the Riemann-Liouville fractional integral of order v as

$$_{c}D_{x}^{-v}f(x)=\frac{1}{\Gamma(v)}\int_{c}^{x}(x-t)^{v-1}f(t)\,dt \tag{3.4.1}$$

Proof

Consider the n^{th} order differential equation with the given initial conditions:

$$\begin{aligned}&y^{n}(x)=f(x)\\&y(c)=0,\ y'(c)=0,\dots,y^{(n-1)}(c)=0\end{aligned} \tag{3.4.2}$$

Using the form of the Cauchy function,

$$H(x,t)=\frac{(x-t)^{n-1}}{(n-1)!} \tag{3.4.3}$$

We claim that the unique solution of (3.4.2) is given by

$$y(x)=\int_{c}^{x}\frac{(x-t)^{n-1}}{(n-1)!}f(t)\,dt \tag{3.4.4}$$

By induction

For $n=1$, we have

$$y'(x)=f(x),\ y(c)=0 \tag{3.4.5}$$

Solving (3.4.5) we obtained

$$\int_{c}^{x}y'(t)\,dt=\int_{c}^{x}\frac{(x-t)^{1-1}}{(1-1)!}f(t)\,dt$$

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Since $y(c)=0$, we have

$$y(x)=\int_c^x f(t)\,dt \tag{3.4.6}$$

Now, assume that (3.4.4) is true for n, we show that the equation is also true for $n+1$.

Consider

$$\begin{aligned} &y^{(n+1)}(x)=f(x) \\ &y(c)=0,\ y'(c)=0,\dots,y^{n}(c)=0 \end{aligned} \tag{3.4.7}$$

Since $y^{(n+1)}(x)=(y')^{(n)}(x)$. Let $u(x)=y'(x)$. Then (3.4.7) becomes

$$\begin{aligned} &u^{n}(x)=f(x) \\ &u(c)=0,\ u'(c)=0,\dots,u^{(n-1)}(c)=0 \end{aligned} \tag{3.4.8}$$

Using the induction hypothesis, we noticed that

$$\int_c^x y'(t)\,dt=\int_{z=c}^{x}\left(\int_{t=c}^{z}\frac{(z-t)^{n-1}}{(n-1)!}f(t)\,dt\right)dz$$

$$y(x)-y(c)=\int_{t=c}^{x}\left(\int_{z=t}^{x}\frac{(z-t)^{n-1}}{(n-1)!}f(t)\,dt\right)dz=\int_c^x\frac{(x-t)^{n}}{n!}f(t)\,dt \tag{3.4.9}$$

Since $y(c)=0$, then

$$y(x)=\int_c^x\frac{(x-t)^{n}}{n!}f(t)\,dt \tag{3.4.10}$$

So, (3.4.4) is true

Thus, since $f(x)$ in (3.4.2) is the n^{th} derivative of $y(x)$, we may interpret $y(x)$ as the n^{th} integral of $f(x)$. Therefore,

$${}_cD_x^{-n}f(x)=\frac{1}{(n-1)!}\int_c^x(x-t)^{n-1}f(t)\,dt \tag{3.4.11}$$

Lamma 3.5. The one and two parameter representation of Mittag-Leffler function can be defined in terms of a power series as

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$$E_\alpha(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)},\quad \alpha>0 \tag{3.5.1}$$

$$E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)},\quad \alpha>0,\beta>0 \tag{3.5.2}$$

The exponential series defined by (3.5.2) is a generalisation of (3.5.1). The following relationship holds from the result of the definition of (3.5.2)

$$E_{\alpha,\beta}(x)=\frac{1}{\Gamma(\beta)}+xE_{\alpha,\alpha+\beta}(x) \tag{3.5.3}$$

and

$$E_{\alpha,\beta}(x)=\beta E_{\alpha,\beta+1}(x)+\alpha x\frac{d}{dx}E_{\alpha,\beta+1}(x) \tag{3.5.4}$$

$$\Rightarrow \frac{d}{dx}E_{\alpha,\beta+1}(x)=\frac{1}{\alpha x}\left[E_{\alpha,\beta-1}(x)-(\beta-1)E_{\alpha,\beta+1}(x)\right] \tag{3.5.5}$$

So that

$$\frac{d}{dx}E_{\alpha,\beta}(x)=\frac{1}{\alpha x}\left[E_{\alpha,\beta-1}(x)-(\beta-1)E_{\alpha,\beta}(x)\right] \tag{3.5.6}$$

Proof of (3.5.3)

By definition (3.5.2), we have that

$$E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)}=\sum_{k=1}^{\infty}\frac{x^{k+1}}{\Gamma(\alpha(k+1)+\beta)}=\sum_{k=1}^{\infty}\frac{xx^k}{\Gamma(\alpha k+(\alpha+\beta))}$$
$$=\frac{1}{\Gamma(\beta)}+x\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+(\alpha+\beta))}=\frac{1}{\Gamma(\beta)}+xE_{\alpha,\alpha+\beta}(x) \tag{3.5.7}$$

Observe that $E_{\alpha,\beta}(0)=1$. Also for specific values of α and β the Mittag-Leffler function reduces to some familiar functions such as

$$E_{1,1}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{x^k}{k!}=\exp(x) \tag{3.5.8}$$

$$E_{\frac{1}{2},1}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k/2+1)}=\exp(x^2)Erfc(-x) \tag{3.5.9}$$

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$$E_{1,2}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+2)}=\sum_{k=0}^{\infty}\frac{x^{k+1}}{(k+1)!}=\frac{\exp(x)-1}{x} \quad (3.5.10)$$

Proposition 3.6. The Gamma function has some unique properties. By the use of its recursion relations, one can obtain the formulas

$$\Gamma(x+1)=x\Gamma(x),\quad x\in\mathbb{R}^+ \quad (3.6.1)$$

$$\Gamma(x)=(x-1)!,\quad x\in\mathbb{N} \quad (3.6.2)$$

From (3.6.1), we observe that $\Gamma(1)=1$. We show that $\Gamma(1/2)=\sqrt{\pi}$

From the definition of Gamma function $\Gamma(x)=\int_0^{\infty}e^{-t}t^{x-1}dt,\ x\in\mathbb{R}^+$

We have

$$\Gamma(1/2)=\int_0^{\infty}e^{-t}t^{1/2-1}dt=\int_0^{\infty}e^{-t}t^{-1/2}dt \quad (3.6.3)$$

If we let $t=y^2\Rightarrow dt=2ydy$, so that

$$\Gamma(1/2)=2\int_0^{\infty}e^{-y^2}dy \quad (3.6.4)$$

Equally, we can write (3.6.4) as

$$\Gamma(1/2)=2\int_0^{\infty}e^{-x^2}dx \quad (3.6.5)$$

Multiplying both (3.6.4) and (3.6.5) together to get

$$\left[\Gamma(1/2)\right]^2=4\int_0^{\infty}\int_0^{\infty}e^{-\left(x^2+y^2\right)}dxdy \quad (3.6.6)$$

Equation (3.6.6) is a double integral and can be in polar coordinates to get

$$\left[\Gamma(1/2)\right]^2=4\int_0^{\pi/2}\int_0^{\infty}e^{-r^2}rdrd\theta=\pi \quad (3.6.7)$$

So that $\Gamma(1/2)=\sqrt{\pi}$

Proposition 3.7. let f be a continuous function and for any $\alpha>0,\ n-1<\alpha\le n$. Then for any positive integer k, we have

$$L\left[{}_{0}^{c}D_x^{\alpha}f(x)\right]=S^{\alpha}F(s)-\sum_{k=0}^{n-1}S^{\alpha-k-1}f^{(k)}(0) \quad (3.7.1)$$

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Proof

recall that in the integer order operations, the Laplace transform of $f^{(n)}$ is given by

$$L\left\{f^{(n)}\right\} = S^n F - S^{n-1} f(0) - S^{n-2} f'(0) - \ldots - f^{n-1}(0)$$
$$= S^n F(s) - \sum_{k=0}^{n-1} S^{n-k-1} f^k(0) \quad (3.7.2)$$

let $f^{(n)}(x) = g(x)$ so that $L\left[{}_0\mathrm{I}_x^{n-\alpha} g(x)\right] = S^{-(n-\alpha)} G(s)$

where,

$$G(s) = L\{g(x)\} = L\left\{f^{(n)}(x)\right\} = S^n F(s) - \sum_{k=0}^{n-1} S^{n-k-1} f^{(k)}(0) \quad (3.7.2)$$

Example 3.8. Consider the initial value problem of the form

$$y^{\alpha}(t) + 2\eta w_m y^{\alpha-1}(t) + w_m^2 y(t) = f(t)$$
$$y'(0) = y_0', y(0) = \mathrm{y}_0 \text{ and } \alpha \geq 2 \quad (3.8.1)$$

Taking the Fourier transform of (3.8.1), gives

$$\int_0^{\infty} \left(y''(t) + 2\eta w_m y'(t) + w_m^2 y(t)\right) \exp(-i\omega t)\, dt = \int_0^{\infty} f(t) \exp(-i\omega t)\, dt \quad (3.8.2)$$

Where,

$$\int_0^{\infty} y'(t) \exp(-i\omega t)\, dt = \left[y(t) \exp(-i\omega t)\right]_0^{\infty} - \int_0^{\infty} -i\omega y(t) \exp(-i\omega t)\, dt$$
$$= -y_0 + i\omega \int y(t) \exp(-i\omega t)\, dt \quad (3.8.3)$$

Noting that

$$\lim_{t \to \infty} y(t) = 0$$

$$\Rightarrow \int_0^{\infty} y''(t) \exp(-i\omega t)\, dt = \left[\exp(-i\omega t) y'(t)\right]_0^{\infty} - \int_0^{\infty} -i\omega y'(t) \exp(-i\omega t)\, dt$$
$$-y_0' + i\omega y_0 + (i\omega)^2 \int_0^{\infty} y(t) \exp(-i\omega t)\, dt \quad (3.8.4)$$

Substituting, into (3.8.1), gives

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$$-y_0' + i\omega y_0 + (i\omega)^2 \int_0^\infty y(t)\exp(-i\omega t)\,dt - 2\eta w_m y_0 + 2\eta w_m i\omega \int_0^\infty y(t)\exp(-i\omega t)\,dt$$
$$+w_m^2 \int_0^\infty y(t)\exp(-i\omega t) = \int_0^\infty f(t)\exp(-i\omega t)\,dt \qquad (3.8.5)$$

$$\Rightarrow \left[(i\omega)^2 + 2i\eta w_m + w_m^2\right]\int_0^\infty y(t)\exp(-i\omega t)\,dt = \int_0^\infty f(t)\exp(-i\omega t)\,dt + \left(y_0' + 2\eta w_m y_0 + i\omega y_0\right)$$
$$\int_0^\infty y(t)\exp(-i\omega t)\,dt = \frac{\int_0^\infty f(t)\exp(-i\omega t)\,dt}{(i\omega)^2 + 2i\eta w_m + w_m^2} + \frac{y_0' + 2\eta w_m y_0 + i\omega y_0}{(i\omega)^2 + 2i\eta w_m + w_m^2} \qquad (3.8.6)$$

Integrating (3.8.6) from $-\infty$ to ∞ on the left hand side to get

$$\int_{-\infty}^\infty \ldots \exp(i\omega\tau)\frac{d\omega}{2\pi} \to \int_{-\infty}^\infty \exp(i\omega(\tau - t))\,d\omega = 2\pi\delta(\tau - t) \to \int_0^\infty y(t)\delta(t-\tau)\,dt \qquad (3.8.7)$$

On the right hand side, we have

$$\int_0^\infty \frac{f(t)}{2\pi}\left(\int_{-\infty}^\infty \frac{\exp(i\omega(\tau - t))\,d\omega}{(i\omega)^2 + 2i\eta w_m w + w_m^2}\right)dt + \int_{-\infty}^\infty \frac{y_0' + 2\eta w_m y_0 + i\omega y_0}{(i\omega)^2 + 2i\eta w_m w + w_m^2}.\frac{\exp(i\omega\tau)\,d\omega}{2\pi} \qquad (3.8.8)$$

Recall the Residue Theorem (By Contour Integral) and using the fact that

$$d\omega = \frac{1}{i}d(i\omega) = s = i\omega$$

Equation (3.8.8) becomes

$$\frac{1}{i}\oint \frac{\exp(s(\tau - t))}{s^2 + 2\eta w_m s + w_m^2}\,ds + \frac{1}{i}\oint \frac{y_0' + 2\eta w_m y_0 + i\omega y_0}{s^2 + 2\eta w_m s + w_m^2}\exp(s\tau)\,ds \qquad (3.8.9)$$

Finding the poles of the denominator

Let $s^2 + 2\eta w_m s + w_m^2 = 0 \to s = -\eta w_m \pm i\sqrt{1-\eta^2}.w_m = -\eta w_m \pm i w_d w_m$

Where, η^2 is damping coefficient and $\eta \in [0,1]$, $w_d \in \mathbb{R}^+$

By Residue theorem,

Article DOI: 10.52589/AJSTE-3EVF5SPX
DOI URL: https://doi.org/10.52589/AJSTE-3EVF5SPX

$$\frac{1}{i}\oint \frac{\exp(s(t-\tau))}{s^2+2\eta w_m s+w_m^2}ds=\int_{-\infty}^{\infty}\frac{\exp(i\omega(\tau-t))}{(i\omega)^2+2i\eta w_m w+w_m^2}ds=\frac{2\pi i}{i}\left[\frac{\exp(s(\tau-t))(s+\eta w_m+iw_d)}{(s+\eta w_m-iw_d)(s+\eta w_m+iw_d)}\right]_{s=\eta sw-iw_d}$$

$$+\left[\frac{\exp(s(\tau-t))}{s+\eta w_m+iw_d}\right]_{s=-\eta sw+iw_d}.\theta(\tau-t)=2\pi\left(\frac{\exp(-\eta w_m(\tau-t))}{2iw_d}+\frac{\exp(iw_d(\tau-t))}{-2iw_d}\right).\theta(\tau-t)$$

$$=\frac{2\pi}{w_d}\exp(-\delta w_m(\tau-t))\sin(w_d(\tau-t)).\theta(\tau-t)$$

Also,

$$\frac{1}{i}\oint \frac{y_0'+2\eta w_m y_0+i\omega y_0}{s^2+2\eta w_m s+w_m^2}\exp(s\tau)ds=\frac{2\pi i}{i}\left(\frac{y_0'+2\eta w_m y_0+(-\eta w_m-iw_d)}{-2iw_d}\right)y_0\exp(-\eta w_m\tau).\exp(iw_d\tau)$$

$$+\frac{y_0'+2\eta w_m y_0+(-\eta w_m+iw_d)y_0}{2iw_d}\exp(-\eta w_m\tau)\exp(-iw_d\tau)\theta(\tau)$$

$$=-\frac{2\pi\exp(-\eta w_d\tau)}{w_d}\left[(y_0'+\eta w_m y_0)\sin w_d\tau-w_d y_0\cos(w_d\tau)\right]\theta(\tau)$$

$$=\frac{-2\pi\exp(-\eta w_m\tau)}{w_d}\sqrt{(y_0'+\eta w_m y_0)^2+(w_d y_0)^2}\cos(w_d\tau+\varphi)\theta(\tau)$$

where,

$$\cos\varphi=-\frac{w_d y_0}{\sqrt{(y_0'+\eta w_m y_0)^2+(w_d y_0)^2}},\quad \sin\varphi=-\frac{y_0'+\eta w_m y_0}{\sqrt{(y_0'+\eta w_m y_0)^2+(w_d y_0)^2}},$$

$\theta(\tau-t)$ is a step function given by $\theta(\tau-t)=\begin{cases}1, & \text{if } \tau>t\\ 0, & \text{otherwise}\end{cases}$

$$\therefore y(\tau)\theta(\tau)\to\int_0^{\infty}y(t)\delta(t-\tau)dt=y(\tau)\theta(\tau)=\int_0^{\infty}f(t)\frac{\exp(-\eta w_m(\tau-t))}{w_d}\sin(w_d(\tau-t))\theta(\tau-t)dt$$

$$-y_0\exp(-\eta w_m\tau)\sqrt{\left(\frac{y_0'+\eta w_m y_0}{w_d}\right)^2}\cos(w_d\tau+\varphi)\theta(\tau)$$

$$\tau>0: y(\tau)=\int_0^{\tau}f(t)\frac{\exp(-\eta w_m(\tau-t))}{w_d}\sin(w_d(\tau-t))dt-\frac{y_0}{|\cos\varphi|}\exp(-\eta w_m\tau)\cos(w_d+\varphi)$$

Where,

$$\tau>0: y(\tau)=\int_0^{\tau}f(t)\frac{\exp(-\eta w_m(\tau-t))}{w_d}\sin(w_d(\tau-t))dt$$

is a solution due to the function $f(t)$ which makes the equation non-homogeneous and

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$$\tau > 0 : y(\tau) = -\frac{y_0}{|\cos\varphi|} \exp(-\eta w_m \tau) \cos(w_d + \varphi)$$

is a solution to the ODE satisfied with the initial condition

CONCLUSION

Fractional Fourier transform is a generalisation of the ordinary Fourier transform. Unlike the other integer order calculus where operations are centred mainly at the integers, fractional calculus considers every real positive number. This work focuses on understanding the properties of fractional derivatives and their effectiveness in certain complex variables, while also constructing non-homogeneous fractional differential equations using the Fourier transform method.

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DOI URL: https://doi.org/10.52589/AJSTE-3EVF5SPX