Advanced Journal of Science, Technology and Engineering
ISSN: 2997-5972
Volume 4, Issue 2, 2024 (pp. 136-144)
www.abjournals.org

COMPARISON ANALYSIS OF METHODS OF ESTIMATION: A NON-BAYESIAN ESTIMATION OF MARSHAL OLKIN ALPHA POWER INVERSE EXPONENTIAL DISTRIBUTION

Adegbite Ismaila Olawale[1*], Asabi Oladipupo[1], Omisore Adedotun Olurin[1],

and Adewoye Kamorudeen Sola[2]

[1]Department of Statistics, Osun State Polytechnic, Iree, Nigeria.

[2]Department of Science and Social Sciences, Directorate of General Studies, Osun State Polytechnic, Iree, Nigeria.

[*]Corresponding Author's Email: adegbiteonline@gmail.com, Tel.: +2347039625838

Cite this article:

Adegbite, I. O., Asabi, O., Omisore, A. O., Adewoye, K. S. (2024), Comparison Analysis of Methods of Estimation: A Non-Bayesian Estimation of Marshal Olkin Alpha Power Inverse Exponential Distribution. Advanced Journal of Science, Technology and Engineering 4(2), 136-144. DOI: 10.52589/AJSTE-2KE34JHJ

Manuscript History

Received: 30 May 2024

Accepted: 4 Aug 2024

Published: 12 Aug 2024

ABSTRACT: *A non-Bayesian approach to parameter estimation, statistical inference and decision-making are discussed and compared. A pragmatic criterion, success in practice, as well as logical consistency is emphasized in comparing alternative approaches. In this study, attention is given to skew distribution for modelling lifetime data in particular: the Marshall Olkin Alpha Power Inverse Exponential (MOAPIE) distribution. Parameters of the distribution were estimated using non-Bayesian estimation methods of Maximum Likelihood Estimation, Least Square Estimation and Weighted Least Square Estimation. Finally, simulated and real life data applications illustrate the performance of the estimation methods.*

KEYWORDS: Non-Bayesian estimation, Maximum Likelihood estimation, Least Square estimation, Weighted Least Square estimation, Simulation.

INTRODUCTION

The non-Bayesian estimation somehow ignores what we know about the situation and just gives you a yes or no answer about trusting the null hypothesis, based on a fairly arbitrary cutoff. In Bayesian inference, a predictive distribution for future data is derived by integrating out unknown parameters; integrating over the posterior distribution of those parameters gives a posterior predictive distribution—a distribution for future data conditional on those already observed. Non-Bayesian methods for predictive inference take into account uncertainty in parameter estimates (Bjornstad, 1990). Non-Bayesian predictive inference is a relatively recent field. Under the heading of "non-Bayesian," we can subdivide the approaches into those that are "classical" frequentist and those that are "likelihood" based (Bjornstad, 1990).

(i) Classical Frequentist Prediction

The focus in frequentism is to achieve the nominal coverage under repeated sampling. Frequentist inference insists on a sharp distinction between unobserved, but nonrandom "parameters" and observable, random data. It works entirely with probability distributions of data, conditional on unknown parameters. It considers the random behavior of functions of the data, estimators, test statistics and makes assertions about the distributions of those functions of the data.

(ii) Likelihood-based Inference

Likelihood-based approaches, like many modern statistical concepts, can be traced back to Ronald Fisher. The basic idea of this school is that, except for special cases, our statistical inferences are on logically weaker grounds than when we are dealing with inferences from a normal distribution whose parameter estimates are orthogonal (Yudi, 1990). Therefore, we can see likelihood as akin to Bayesian probability, but without the integrability requirements or the possible confusion with frequentist probability.

Bayesian inference treats everything as random before it is observed, and everything observed as "once observed" is treated as no longer random (Basheer et al., 2021). It aims at assisting in constructing probability statements about anything as yet unobserved (including "parameters") conditional on the observed data. HajAhmad and Almetwally (2020) stated that Bayesian inference aids in making assertions like, "given the observed data." Bayesian inference therefore feeds naturally into discussion of decisions that must be made under uncertainty, while frequentist analysis does not.

In this study, attention is given to a non-normal distribution for modelling lifetime data in particular: the Marshall Olkin Alpha Power Inverse Exponential (MOAPIE) distribution formulated by Basheer (2019). Parameters of the distribution will be estimated using non-Bayesian estimation methods of Maximum Likelihood Estimation, Least Square Estimation and Weighted Least Square Estimation and thereafter the comparison of the approaches.

Article DOI: 10.52589/AJSTE-2KE34JHJ
DOI URL: https://doi.org/10.52589/AJSTE-2KE34JHJ

Advanced Journal of Science, Technology and Engineering
ISSN: 2997-5972
Volume 4, Issue 2, 2024 (pp. 136-144)

THE MODEL

In this section, we present the MOAPIE distribution about which some non-Bayesian estimation technique of estimation parameter will be carried upon.

The cumulative distribution function (CDF) of the MOAPIR is

$$G_{MOAPIR}(x;\alpha,\lambda,\theta) = \begin{cases} \dfrac{\alpha^{e^{-\lambda x^{-1}}}-1}{\theta(\alpha-1)+(1-\theta)\left(\alpha^{e^{-\lambda x^{-1}}}-1\right)}, & x>0,\alpha\neq 1,\lambda>0,\theta>0 \\ e^{-\lambda x^{-1}}, & \alpha = 1 \end{cases} \quad (1)$$

Where α, λ and θ are shape, scale and location parameters respectively. (Basheer, 2019)

The probability density function (PDF)

$$g_{MOAPIR}(x;\alpha,\lambda,\theta) = \begin{cases} \dfrac{(\alpha-1)2\lambda\theta\log(\alpha)x^{-3}e^{-\lambda x^{-2}}\alpha^{e^{-\lambda x^{-1}}}}{\left[(\alpha-1)\theta+(1-\theta)\left(\alpha^{e^{-\lambda x^{-1}}}-1\right)\right]^2} & x>0,\alpha\neq 1,\lambda>0,\theta>0 \\ \lambda x^{-2}e^{-\lambda x^{-1}}, & \alpha = 1 \end{cases} \quad (2)$$

(Basheer, 2019).

METHODS OF ESTIMATION

For the considered distribution, we adopted three methods of estimation of non-Bayesian estimation (Maximum likelihood, Least square and Weighted least square) for estimating the parameters of MOAPIE distribution.

Maximum Likelihood Estimation

Let $x_1,\dots,x_n$ be a random sample from MOAPIE $=(\alpha,,)$ distribution; then the likelihood function is given by

$$L(x|\Theta) = \prod_{i=1}^{n} g(x_i) = \frac{(\alpha-1)^{n}(\log(\alpha))^{n}\lambda^{n}\theta^{n}e^{-\lambda\sum_{i=1}^{n}x_i^{-1}}\alpha^{\sum_{i=1}^{n}e^{-\lambda x_i^{-1}}}\prod_{i=1}^{n}x_i^{-2}}{\prod_{i=1}^{n}\left[(\alpha-1)\theta+(1-\theta)\left(\alpha^{e^{-\lambda x_i^{-1}}}-1\right)\right]^2} \quad (3)$$

By taking logarithm of the likelihood function, we have

Article DOI: 10.52589/AJSTE-2KE34JHJ
DOI URL: https://doi.org/10.52589/AJSTE-2KE34JHJ

$$\ell(x|\Theta) = nlog\left((\alpha-1)\log(\alpha)\,\lambda\theta\right) - \lambda\sum_{i=1}^{n} x_i^{-1} + \log(\alpha)\sum_{i=1}^{n} e^{-\lambda x_i^{-1}} - 2\sum_{i=1}^{n}\log(x_i) - 2\sum_{i=1}^{n}\log\left[(\alpha-1)\theta + (1-\theta)\left(\alpha^{e^{-\lambda x_i^{-1}}} - 1\right)\right] \quad (4)$$

To obtain the MLEs of α, λ and θ, we differentiate the expression in Eq. (4) with respect to α, λ and θ. Thus, we have

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha-1} + \frac{n}{\alpha\log(\alpha)} + \frac{1}{\alpha}\sum_{i=1}^{n} e^{-\lambda x_i^{-1}} - 2\sum_{i=1}^{n}\frac{\theta + (1-\theta)e^{-\lambda x_i^{-1}}\alpha^{e^{-\lambda x_i^{-1}}} - 1}{(\alpha-1)\theta + (1-\theta)(\alpha^{e^{-\lambda x_i^{-1}}} - 1)} \quad (5)$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i-1}^{n} x_i^{-1} - \log(\alpha)\sum_{i=1}^{n} x_i^{-1}e^{-\lambda x_i^{-1}} - 2\sum_{i=1}^{n}\frac{(1-\theta)\log(\alpha)x_i^{-1}e^{-\lambda x_i^{-2}}\alpha^{e^{-\lambda x_i^{-1}}}}{(\alpha-1)\theta + (1-\theta)(\alpha^{e^{-\lambda x_i^{-1}}} - 1)} \quad (6)$$

and

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} - 2\sum_{i-1}^{n}\frac{\alpha - \alpha^{e^{-\lambda x_i^{-1}}}}{(\alpha-1)\theta + (1-\theta)(\alpha^{e^{-\lambda x_i^{-1}}} - 1)} \quad (7)$$

On equating equations 5, 6 and 7 to zero, we obtained the MLE of α, λ, and θ. These equations cannot be solved analytically. Hence iterative methods such as Newton-Raphson can be used to accomplish the task of estimating the parameters (Almetwally et al, 2021).

Article DOI: 10.52589/AJSTE-2KE34JHJ
DOI URL: https://doi.org/10.52589/AJSTE-2KE34JHJ

Least Square Estimation

Swain and Wilson (1988) introduced the Least Square Estimators (LSE). We use the least square procedure for estimating the parameters α, λ, and θ of the MOAPIE distribution. The least square estimation is obtained by minimizing

$$P(\alpha,\lambda,\theta)=\sum_{i=1}^{n}\left(G(X_i,\Theta)-\frac{i}{n+1}\right)^2 \tag{8}$$

After differentiating the equation (8) with respect to parameters Θ and then equating to zero

$$\frac{\partial P_{LS}}{\partial\alpha}=2\sum_{i=1}^{n}\left(G(x_i,\Theta)-\frac{i}{n+1}\right)\left[\frac{\alpha^{e^{-\lambda x_i^{-1}}}-1\ e^{-\lambda x_i^{-1}}}{\theta(\alpha-1)+(1-\theta)\left(\alpha^{e^{-\lambda x_i^{-1}}}-1\right)} - \frac{G(x_i,\Theta)(\theta-\alpha^{e^{-\lambda x_i^{-1}}}\ e^{-\lambda x_i^{-1}}(\theta-1))}{\theta(\alpha-1)+(1-\theta)\left(\alpha^{e^{-\lambda x_i^{-1}}}-1\right)}\right] \tag{9}$$

$$\frac{\partial P_{LS}}{\partial\lambda}=2\sum_{i=1}^{n}\left(G(x_i,\Theta) - \frac{i}{n+1}\right)\left[\frac{-\alpha^{e^{-\lambda x_i^{-1}}}\ e^{-\lambda x_i^{-1}}x_i^{-1}\log(\alpha)\,(1+(\theta-1)G(x_i,\Theta))}{\theta(\alpha-1)+(1-\theta)\left(\alpha^{e^{-\lambda x_i^{-1}}}-1\right)}\right] \tag{10}$$

and

$$\frac{\partial P_{LS}}{\partial\theta}=2\sum_{i=1}^{n}\left(G(x_i,\Theta)-\frac{i}{n+1}\right)G(x_i,\Theta)\,(-\alpha+\alpha^{e^{-\lambda x_i^{-1}}}) \tag{11}$$

The above stated non-linear equations cannot be solved analytically, so the $\widehat{\Theta}_{LS}\ of\ \Theta$ can use any iterative procedure techniques, such as conjugate-gradient algorithms, to obtain the numerical solution.

Article DOI: 10.52589/AJSTE-2KE34JHJ
DOI URL: https://doi.org/10.52589/AJSTE-2KE34JHJ

Weighted Least Square Estimation

Weighted Least Square Estimation (WLSE) was also introduced by Swain and Wilson (1988). We adopted the WLS procedure to estimate parameters α, λ, and θ of the MOAPIE distribution. The WLSE estimation is obtained by minimizing

$$W(\alpha,\lambda,\theta) = \sum_{i=1}^{n} W_i \left(G(X_i,\Theta) - \frac{i}{n+1} \right)^2 \tag{12}$$

Where

$$W_i = \frac{(n+1)^2(n+2)}{i(n-i+1)} \tag{13}$$

After differentiating the equation (12) with respect to Θ and then equating to zero

$$\frac{\partial W}{\partial \alpha} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left(G(x_i,\Theta) - \frac{i}{n+1} \right) \left[\frac{\alpha^{e^{-\lambda x_i^{-1}}} - 1 \ e^{-\lambda x_i^{-1}}}{\theta(\alpha-1) + (1-\theta)\left(\alpha^{e^{-\lambda x_i^{-1}}} - 1\right)} - \frac{G(x_i,\Theta)(\theta - \alpha^{e^{-\lambda x_i^{-1}}} \ e^{-\lambda x_i^{-1}}(\theta-1))}{\theta(\alpha-1) + (1-\theta)\left(\alpha^{e^{-\lambda x_i^{-1}}} - 1\right)} \right] \tag{14}$$

$$\frac{\partial P_{LS}}{\partial \lambda} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left(G(x_i,\Theta) - \frac{i}{n+1} \right) \left[\frac{-\alpha^{e^{-\lambda x_i^{-1}}} \ e^{-\lambda x_i^{-1}} x_i^{-1} \log(\alpha)\,(1 + (\theta-1)G(x_i,\Theta))}{\theta(\alpha-1) + (1-\theta)\left(\alpha^{e^{-\lambda x_i^{-1}}} - 1\right)} \right] \tag{15}$$

and

$$\frac{\partial P_{LS}}{\partial \theta} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left(G(x_i,\Theta) - \frac{i}{n+1} \right) G(x_i,\Theta)\left(-\alpha + \alpha^{e^{-\lambda x_i^{-1}}}\right) \tag{16}$$

Article DOI: 10.52589/AJSTE-2KE34JHJ
DOI URL: https://doi.org/10.52589/AJSTE-2KE34JHJ

Advanced Journal of Science, Technology and Engineering
ISSN: 2997-5972
Volume 4, Issue 2, 2024 (pp. 136-144)
www.abjournals.org

SIMULATION

To compare between the Maximum Likelihood Estimation, Least Square Estimation and Weighted Least Square Estimation methods, the parameters of MOAPIE distribution are estimated using Monte Carlo Simulation, which is implemented by R language. The data were generated from the MOAPIE distribution for lifetime of arbitrary value of parameters α, λ, and θ α=1.5, λ=1.5, and θ=1.6.

Table 1: Bias and Mean Square Error (MSE) of Parameters α=1.5, λ=1.5, and θ=1.6

		MLE	LSE	WLSE
n	Parameters	Bias MSE	Bias MSE	Bias MSE
50	α	0.1526 0.6731	0.0432 0.2310	0.0241 0.0745
	λ	0.2521 0.8745	0.2008 0.4035	0.1112 0.3425
	θ	0.5320 2.4287	0.0353 0.6534	0.0231 0.3421
100	α	0.1134 0.3248	0.0431 0.0922	0.0265 0.0542
	λ	0.1154 0.3215	0.0433 0.1232	0.0439 0.1290
	θ	0.4295 1.2401	0.0145 0.0865	0.0132 0.2973
200	α	0.0834 0.2390	0.0438 0.0645	0.0245 0.0432
	λ	0.0354 0.2315	0.0345 0.0947	0.0132 0.0653
	θ	0.3310 0.6476	0.0437 0.243	0.0243 0.1076

The Bias and MSE are computed by generating 1000 replications samples size n=50, 100 and 200 from the MOAPIE distribution. Table 1 reveals that for each method, the biases and the MSE's decrease as sample size n increases. Also, the Weighted Lest Square estimation is the best estimation among the three considered Non-Bayesian estimations.

Article DOI: 10.52589/AJSTE-2KE34JHJ
DOI URL: https://doi.org/10.52589/AJSTE-2KE34JHJ

FITTING RELIABILITY DATA

Numerical results of the parameter estimation of MOAPIE distribution are presented here with two real data.

Data Set 1: The data set was obtained from Lawless (2011) which consists of the failure time for 17 appliances. The data are: 1167, 1925, 1990, 2223, 2400, 2471, 2551, 2568, 2694, 3034, 3112, 3214, 3478, 3504, 4329, 6976, 7846.

Parameters of MOAPIE distribution were estimated using MLE, LSE and WLSE. Kolmogorov-Smirnov (K-S) distance and p-values were also computed.

Table 2: Estimated and Standard Error (Std) of Parameters for Different Methods of Estimation

Parameters		MLE	LSE	WLSE
α	Estimate	223.42	219.22	465.81
	Std	2.54×10^{-5}	6.34×10^{-5}	2.01×10^{-6}
θ	Estimate	542.67	561.34	1529.11
	Std	8.34×10^{-5}	1.23×10^{-5}	4.76×10^{-7}
λ	Estimate	525.01	531.89	1634.97
	Std	7.42×10^{-6}	3.49×10^{-5}	5.76×10^{-7}
K-S		0.1865	0.1852	0.1852
P-value		0.2321	0.2302	0.2302

Data Set 2: The data set was obtained from Gacula and Kubala (1975). It consists of failure times of a certain product and contains 26 observations: 24, 24, 26, 32, 32, 33, 33, 33, 35, 41, 42, 43, 47, 48, 48, 48, 50, 52, 54, 55, 57, 57, 57, 57, 61.

The MLE, LSE and WLSE estimates of the parameters and the values of the K-S statistic with p-values are presented in Table 3.

Table 3: Estimated and Standard Error (Std) of Parameters for Different Methods of Estimation

Parameters		MLE	LSE	WLSE
α	Estimate	254.72	259.87	501.72
	Std	1.89×10^{-5}	2.73×10^{-5}	2.52×10^{-6}
θ	Estimate	643.87	684.78	1256.34
	Std	8.34×10^{-5}	1.23×10^{-5}	4.76×10^{-7}

Article DOI: 10.52589/AJSTE-2KE34JHJ
DOI URL: https://doi.org/10.52589/AJSTE-2KE34JHJ

λ	Estimate	550.65	583.09	129.03
	Std	7.42×10^{-6}	3.49×10^{-5}	5.76×10^{-7}
K-S		0.178	0.1813	0.1813
P-value		0.3012	0.3002	0.3002

In selecting the best estimation method, the values of K-S, P-values, Standard error and Mean Square error were computed for the MOAPIE distribution. From Table 3, since WLSE has the minimum value of Bias, MSE and Std, Weighted Least Square is adjudged to be the best estimation method among the three methods of estimation considered.

CONCLUSION

In this paper, we considered three different approaches under non-Bayesian estimation method to estimate the unknown parameters of the Marshall Olkin Alpha Power Inverse Exponential distribution (MOAPIE) and provided some applications in the context of statistics. The results were explicated and revealed that Weighted Least Square provides a best fit and choice over the other two considered non-Bayesian estimations (Maximum Likelihood Estimation and Least Square Estimation).

REFERENCES

Almetwally, E.M., Ahmed Z., Afify, A.Z, & Hamedani, G.G. (2021). Marshall-Olkin Alpha Power Rayleigh Distribution: Properties, Characterizations, Estimation and Engineering Applications. *Pak.j.stat.oper.res.* 17(3), 745-760.
Basheer, A.M. (2019). Marshall-Olkin Alpha Power Inverse Exponential Distribution: Properties
And Applications, *Annals of Data Science,* . https://doi.org/10.1007/s40745-019- 00229-0.
Basheer, A.M., Almetwally, E.M. & Okasha, H.M. (2021). Marshall Olkin Alpha Power Inverse Weibull Distribution: Bayesian and Non-Bayesian Estimations. *Journal of Statistics Applications & Probability.* 10 (2), 327-345.
Bjornstad, J. F.(1990). Predictive Likelihood: A Review. *Statist. Sci.* 5(2), 242--254.
Gacula, M.C. & Kubala, J.J. (1975). Statistical Models for Shelf Life Failures, *Journal of Food Science*, 40, 404-409.
Haj Ahmad, H.A., & Almetwally, E. M. (2020). Marshall-Olkin Generalized Pareto Distribution:
Bayesian and Non Bayesian Estimation. *Pak.j.stat.oper.res.* 16(1), 21-33. DOI: http://dx.doi.org/10.18187/pjsor.v16i1.2935.
Lawless, J.F. (2011). Statistical Models and Methods for Lifetime Data, John Wiley & Sons, University of Waterloo.
Swain, J., Venkatraman, S. & Wilson, J. (1988). Least Squares Estimation of Distribution
Function in Johnson’s Translation System, *Journal of Statistical Computation and Simulation,* 29, 271-297.
Yudi P.(2001). In All Likelihood: Statistical Modeling and Inference Using Likelihood. 1st ed. Oxford University Press

Article DOI: 10.52589/AJSTE-2KE34JHJ
DOI URL: https://doi.org/10.52589/AJSTE-2KE34JHJ