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MATHEMATICAL MODELING OF RECOVERY CURVES

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ABSTRACT: *As the population of people in the globe continues to age, there will be a growing demand for health issues including treatment of Total Hip Arthroplasty (THA) and Total Knee Arthroplasty (TKA). This will result in a growth in post-operation rehabilitation services. In order to meet this growth, it will be beneficial for occupational therapists and physical therapists to have an understanding of a patient's expected post-surgery recovery rate. Once developed these rates can be used to benchmark individual patient improvement, help estimate expected costs and lengths of therapy, and possibly help design optimal treatment session scheduling. In this paper, we consider Hierarchical Linear Model to analyze how male and female patients respond to TKA and similar surgeries.*

KEYWORDS: Recovery Curve, Expected Level of Recovery, Intercept of the Recovery Curve, Growth Rate of Recovery Curve, Hierarchical Linear Model, Logistic Regression.

INTRODUCTION

Recovery rates can be modeled as a function whose input is days since surgery and output representing expected recovery status. The graph of such a function provides a visual explanation of recovery rates that can be used to discuss individual patient's recovery. We shall refer to such a graph as a Recovery Curve. In [**1**], [**2**], and [**3**], we find research discussing the form of recovery curves and statistical analysis fitting the recovery curve form to real recovery data collected from various sources. This research provides a solid platform for research on recovery curves and most importantly demonstrates that a Hierarchical Linear Model can be used as a framework to develop recovery curves for post-THA and post-TKA patients. It is likely that such a model would fit recovery curves for other types of surgeries also.

Describing the Model

Research on patient recovery rates has suggested that recovery curves are most likely to take the form of a standard logistic curve, given by $E(d) = \frac{e^{(\alpha+\beta d)}}{1+e^{(\alpha+\beta d)}}$ (2.1) where $E(d)$ represents the expected level of recovery d days after the surgery was performed and the coefficients α and β differ from patient to patient. The coefficient α represents the intercept (or starting point) of the recovery curve, while the coefficient β represents the growth rate of the recovery curve.

The hierarchical portion of the model follows next. Since recovery rate varies from patient to patient, we hope to predict the coefficients α and β through a collection of patient demographics. For example, the coefficient α may be dependent on the patient's gender, age, and weight category. We shall let x represent a vector of patient demographic factors and make the assumption that α and β depend linearly on the vector x. That is, we have the following two equations

$$\alpha = \alpha_0 + \sum_{i=1}^{N} \alpha_i x_i \qquad (2.2)$$

$$\beta = \beta_0 + \sum_{i=1}^{N} \beta_i x_i \qquad (2.3)$$

where x_i ($i = 1, 2, \ldots, N$) are the predictive variables. This provides a two-level model for recovery, as $E(d)$ depends on d (level 1: post-surgery days) and on x (level 2: patient demographic factors).

Notice that under this model, the x_i predictive variables are all treated in the same manner. In particular, if a predictive variable is doubled in value, the effect on the model is doubled. For many predictive variables, such as age, this is not a concern. Basically, it enforces the assumption that as age increases, the impact of age on recovery time increases. However, for predictive variables which are categorical in nature, such as "type of walking aid used", this is a problem. For example, suppose we have three types of walking aids: no aid, walking cane, and wheel chair. If a single numerical variable is used to represent all three types of walking aids, then we would have to give each category a numerical representation. However, this introduces bias into the model, as such a numerical representation automatically assumes that one type of walking aid has a greater effect than another.

Categorizing Predictive Variables

There are two common ways to correct this problem. The more mathematical method is to create a predictive variable for each category. For example, given our three types of walking aids, we create three predictive variables, say x_1, x_2, and x_3, representing the following:

$x_1 = 1$ for no walking aid used

$x_2 = 1$ for walking cane used and

$x_3 = 1$ for wheel chair used

Clearly, for any one patient exactly one of these variables is equal to 1, while the remaining two would be equal to 0. The second, and somewhat easier method, is to use a statistical software package that allows the user to define categorical variables. Categorical variables are variables that have no intrinsic ordering between different variable values. Such variables are treated differently, with the software automatically applying the mathematical method above. If such software is available, then modelers should be careful to define variable types correctly.

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Hierarchical Linear Model

The model created by equations (2.2) and (2.3) is what is referred to as a *hierarchical linear model*, or *multi-level model*. In this case the model consists of two levels, the first being the logistic regression curve for $E(d)$ and the second being the linear regression of α and β. In order to determine the best coefficients for this model, we proceed by reducing it to a *single-level model*. To do this, we construct N new predictive variables defined as $y_i = x_i \times d$ ($i = 1, 2, . . .,N$). Equation (2.1) can now be reduced to a single logistic regression defined by the equation

$$E(d) = \frac{e^{\left(\alpha_0 + \beta_0 d + \sum_{i=1}^{N} \alpha_i x_i + \sum_{i=1}^{N} \beta_i y_i\right)}}{1 + e^{\left(\alpha_0 + \beta_0 d + \sum_{i=1}^{N} \alpha_i x_i + \sum_{i=1}^{N} \beta_i y_i\right)}} \quad (4.1)$$

Thus, the final logistic regression has up to $2N + 2$ undetermined coefficients. Standard statistical software can now be used to determine goodness-of-fit information for the single-level model and determine to which demographics factors are most important in-patient recovery.

Using specific data analysis from **[1]**, **[2]** and **[3]** for the case of TKA, with respect to male and female patients using the model described in equation (4.1), we get the graph as shown in Figure 1 below.

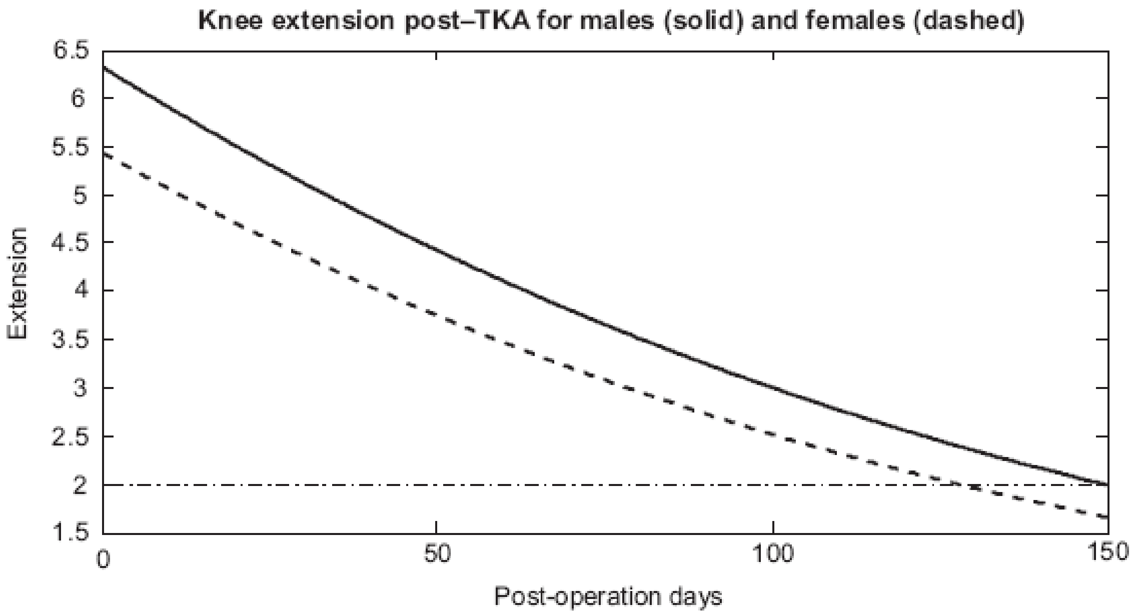

Figure 1: **Post-surgery Knee Extension Recovery Curves**

The expected loss in post-surgery knee extension by post operation date (POD).

CONCLUSION

Using the model described in section 4, it was found that the single largest determining factor of recovery was time past surgery and the type of surgery performed. Gender and the availability of at-home physiotherapy played a small role in certain surgery types but were not significant for most measures of recovery. In Figure 1, we have displayed two sets of sample recovery curves for male and female patients. These examine the expected loss in post-surgery knee extension by post-operation date (POD). The dashed line (at 2 degrees) is the target level of recovery. We notice from the proposed model that the expected recovery rate for males is slightly slower than females. However, the statistical significance of this result is minor. By suitably redefining equation (4.1) we can further improve the nature of recovery curves and thereby know much more than what we had observed in this paper. That will be the scope of further research. Moreover, the Hierarchical Linear Model is found to provide useful information about post surgeries for other surgeries also apart from THA and TKA discussed in this paper.

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