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OPTIMAL TIME CONTROL OF UNIFORM LINEAR MOTION: PROBLEM FORMULATION AND SWITCHING TIME CHARACTERIZATION VIA PONTRYAGIN'S MAXIMUM PRINCIPLE

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ABSTRACT: *Optimal control theory seeks to determine effective controls for dynamical systems, driven by the pressing need for efficient use of energy, time, and resources. This paper examines the optimal time-control problem for the uniform linear motion of a train between two fixed points. The problem is rigorously formulated using differential equations and the Newtonian principles of motion. The Pontryagin Maximum Principle (PMP) is applied to derive the necessary conditions for optimality and to determine the optimal control that minimises the travel time of the body. Both analytical and geometrical approaches are employed to reconstruct optimal trajectories and characterise the switching curve that governs transitions between control values. The study demonstrates that the optimal control is uniquely determined by the initial state of the system relative to the switching curve, providing a practical and computationally tractable feedback control conditions.*

KEYWORDS: Optimal Control, Dynamical System, Uniform Linear Motion, Pontryagin Maximum Principle, Switching Curve.

INTRODUCTION

The famous story of optimal control theory began in 1696 with the brachistochrone problem. In 1696, Johann Bernoulli solved the brachistochrone problem by an ingenious method combining Fermat's principle of minimum time and Snell's laws of refraction (Haws & Kiser, 1995; Parnovsky, 1998; Johnson, 2004). This problem of finding the curve of fastest descent between two points set the stage for centuries of work on optimisation over trajectories (Sussmann and William, 1997).

Addressing issues of dynamic programming, Richard Bellman introduced the principle of optimality and the Hamiltonian-Bellman equation. Bellman's dynamic programming principle represents one of the two cornerstone methodologies in optimal control theory, providing a distinct but complementary approach to deriving optimal controls. These two major tools for studying controlled systems involve the adjoint function, the Hamiltonian function, and the value function; their relationships have been investigated using the notions of superdifferential and subdifferential, connecting the maximum principle to the Hamilton-Jacobi-Bellman equation in the viscosity sense (Bellma, 1957).

The Soviet mathematician Lev Pontryagin published "The Mathematical Theory of Optimal Processes", another cornerstone of the theory. Pontryagin's maximum principle and Bellman's dynamic programming are the two principal approaches in solving optimal control problems, yet they were developed separately and independently. The maximum principle gives necessary conditions for optimality using a Hamiltonian formulation, making it especially powerful for constrained and nonlinear problems (Pontryagin *et al.*, 1962).

Rudolf Kalman's research on contributions to the theory of optimal control extensively transformed engineering practice. With linear quadratic feedback control, and he set the stage for what came to be known as Linear Quadratic Regulator (LQR) control, while the combination with Kalman's filtering paper formed the basis for Linear Quadratic Gaussian (LQG) control (Kalman, 1960).

As surveyed in Bryson (1996), the field matured rapidly through aerospace, economics, and other engineering applications. Today, optimal control theory underpins robotics, autonomous vehicles, financial modelling, and modern artificial intelligence (Bertsekas, 2019; Sahu *et al.,* 2022; Wu, 2025). Also, optimal control has expanded into stochastic systems, model predictive control, and reinforcement learning. The numerical challenges of applying Bellman and Pontryagin's methods to large-scale problems drove new computational approaches. Astrodynamicists, recognising that their elegant methods of Bellman and Pontryagin were not scalable to space trajectory optimisation, developed a broad set of computational tools that frequently required deep physical insights to solve real-world problems.

This paper presents a theoretical background of the Pontryagin maximum principle. A systematic approach to the formulation of the uniform linear problem is considered. The maximum principle is applied to determine the optimal control among a range of controls to find the optimal trajectory to the uniform linear motion problem.

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THEORETICAL UNDERPINNING

Controlled System

We investigate the behaviour of an object whose immediate state is given by $n-$a tuple of real number $(x_1, x_2, \ldots x_N) \in X \subset \mathbb{R}^N$, which X is called a phase space. The values $(x_1, x_2, \ldots x_N)$ are usually coordinates of positions and velocity of an object and they are time-dependent. From the mathematical point of view, there are k values $u_1, \ldots u_k \in U \subset \mathbb{R}^k$ that may influence the motion of the object. The behaviour of the object is assumed to be described by the system of first-order ODEs:

$$\dot{X} = f(x, u) \qquad (1)$$

and the conditions

$$\begin{cases} X(0) = a \\ X(T) = b \end{cases} \qquad (2)$$

where

$X = (x_1, \ldots x_N) \in X, u = (u_1, \ldots, u_k) \in U, f = (f_1, \ldots, f_N) \in N$. The system described by equations (1) and (2) is called a *controlled system.*

Definition 1: A vector variable $u(t) = (u_1(t), \ldots, u_k(t))$ defined on the interval $[t_0, t_1]$ with the valued from $U \subset \mathbb{R}^k$ is called a '*control*' and the set U is called the '*range of control*'. The control problem is characterised by the following key properties:

a. the range of control U is a compact (finite and closed) set in $\mathbb{R}^k$.

b. f satisfies assumptions that ensure the existence and uniqueness of the solution to the controlled system (1) and its initial and terminal conditions in equation (2).

Feasible Control

Definition 2: The set $u = (u_1, \ldots, u_k) \in U$ is considered the class of feasible control if their elements satisfy the following three (3) properties.

P1: If $u(t), t \in I$ is a feasible control, then $t \in J \subset I$ is also a feasible control. Thus, any part of a feasible control is also a feasible control.

P2: If $u_1(t), t \in (t_0, t_1)$ and $u_2(t), t \in (t_1, t_2)$ are feasible controls, then

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$$u(t)=\begin{cases}u_1(t) & t\in(t_0,t_1);\\ u_2(t), & t\in(t_1,t_2)\end{cases}$$

is also a feasible control. Thus, an attachment of two feasible controls is a feasible control.

P3: If $u(t),t\in(t_0,t_1)$ is feasible control, then for any $a\in x$, there exists a unique solution to the initial value problem given in equations (1) and, (2).

Basic Optimal Control Problem

Definition 3: Let $a,b\in X$ be arbitrary among all feasible controls u converting the system from point a to b. We seek a control such that the objective function

$$j=\int_0^T f\left(x(t),u(t)\right)dt \qquad (3)$$

Is minimized just when $u=\hat{u}$ and the finial time $T=\hat{T}$. The control is called an *optimal control,* and the corresponding trajectory is called an *optimal trajectory.* The couple is then $\left(\hat{u}(t),\hat{x}(t)\right),t\in<0,\hat{T}$ is called an *optimal control process.* Mathematically, the optimal control problem is defined as

$$\begin{aligned}&\overset{g}{X}=f(x,u) \quad u\in U\\ &X(T)=a\in X\\ &X(T)=b\in X\\ &j=\int_0^T f\left(x(t),u(t)\right)dt\rightarrow \text{minimise}\end{aligned} \qquad (4)$$

Theorem 1: Let $\hat{u}(t),t\in\left(0,\hat{T}\right)$ be an optimal control converting a to b. Let $\hat{x}(t),t\in\left(0,\hat{T}\right)$ be the corresponding optimal trajectory and $0\le T_1<T_2\le\hat{T}$. Then $\hat{u}(t),t\in<T_1,T_2>$ is an optimal control converting $\hat{x}(T_1)$ to $\hat{x}(T_2)$ and $\hat{x}(t),t(T_1,T_2)$ is the corresponding optimal trajectory. Any part of the optimal control is again optimal control, and the same is true for optimal trajectories.

Pontryagin's Maximum Principle

This section formulates a basic assertion of optimal control theory. For its easier formulation, some additional symbols and variables are introduced. Considering again a basic control system $\overset{g}{X}=f(x,u)$ extended by the additional zero equation $\overset{g}{X}_0=f_0(x,u)$, where $f_0(x,u)$ is an objective function of our problem, yields the *extended controlled problem*

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$x^* = f^*(x,u);$

Where $x^* = (x_0, x_1, ..., x_n), f^* = (f_0 ... f_n)$ and $x = (x_1, ..., x_n)$.

Further introduce the differential system for additional variables $\psi_0.\psi_1....,\psi_n$ in the form

$$\dot{\psi}_0 = -\sum_{k-0}^{n} \frac{\partial f_k(x,u)}{\partial x_0} \psi_k$$
$$\dot{\psi}_1 = -\sum_{k-0}^{n} \frac{\partial f_k(x,u)}{\partial x_1} \psi_k \tag{5}$$
$$\vdots$$
$$\dot{\psi}_n = -\sum_{k-0}^{n} \frac{\partial f_k(x,u)}{\partial x_n} \psi_k .$$

These variables $\psi_0, \psi_1, ..., \psi_n$ play the same role as Lagrange multipliers in extrema of functions of more variables, having the same role as classical multipliers in optimisation problems of functions of several variables. We denote

$\psi^* = (\psi_0.\psi_1....,\psi_n)$ and introduce the Hamiltonian

$$H^* = \psi^* \cdot f^* = \sum_{k=0}^{n} \psi_k f_k . \tag{6}$$

The extended control system and the additional system for the Lagrange multiplier jointly form the Hamiltonian system

$$\begin{cases} \dot{x}_i = \dfrac{\partial H^*}{\partial \psi_i}; \quad i = 0,1,3,...n \quad \text{the extended control} \\ \dot{\psi}_i = -\dfrac{\partial H^*}{\partial x_i}; \; i = 0,1,2,...n \quad \text{the djoint system} \end{cases} \tag{7}$$

Maximum Principle for Time Optimisation Problem

Suppose $\hat{u}(t), t \in \left[0, \hat{T}\right]$ is a solution of the problem (2) and let $\hat{x}(t) \in X$ be the corresponding optimal trajectory. Then, there exists a continuous and non-zero solution $\psi = (\psi_0.\psi_1....,\psi_n)$ of the adjoint system

$$\dot{\psi}_i = \sum_{k=0}^{n} \frac{H^*}{\partial x_i}\left(\psi^*, \hat{x}, \hat{u}\right), \; i = 1,...n \text{ on } \left[0, \hat{T}\right] \tag{8}$$

such that H satisfies the maximum condition

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$$MaxH^*\left(\psi^*, \hat{x}, u\right) = H^*\left(\psi^*, \hat{x}, \hat{u}\right) \tag{9}$$

on $[0,,\hat{T}]$. Moreover, the right-hand side

$$H^*\left(\psi^*, \hat{x}, \hat{u}\right) \equiv 0 \tag{10}$$

on $[0,,\hat{T}]$ and $\psi_0(t) \equiv \psi_0 \leq 0$. The maximum principle is a necessary condition for the optimality of $\hat{u}$. The main part of the assertion is the maximum condition in equation (9). Although this is only a necessary condition, it is strong enough to generate several optimal control candidates and, in many cases, gives a unique candidate for optimal control. For proof of the Pontryagin's Maximum Principle, see Lewis (2006), Schättler and Ledzewicz (2012) and, Tauchnitz (2015).

FORMULATION OF UNIFORM LINEAR MOTION PROBLEM

The uniform linear motion problem is stated as follows: Consider a train moving uniformly without friction along a straight rail under an external drawing force, that can be changed in a given time border. The goal is to find a strategy such that the train stops at a place in a minimum amount of time.

Mathematical Formulation

Let introduce the length of the rail to be covered by coordinate y and choose the required stopping or terminal point of the cart as the origin. Let a_1 be initial position, a_2 initial velocity of the cart of mass m. The drawing force is the control here, and its denoted by u. The dynamics of the moving cart is given by the Newton's law of motion

$$m\ddot{y} = u \tag{11}$$

$u \in (-\alpha, \beta), \alpha, \beta > 0$. The initial conditions are:

$$\begin{cases} y(0) = a_1 \\ \dot{y}(0) = a_2 \end{cases} \tag{12}$$

and terminal condition

$$\begin{cases} y(T) = 0 \\ \dot{y}(T) = 0 \\ T \to \min. \end{cases} \tag{13}$$

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Without loss of generality, we let $m=1$. The Equation (11) become $\ddot{y}=u$ and as such $\alpha=\alpha/m$ and, $\beta=\beta/m$. Transforming the second order ODE to first ODE, let introduce the phase variable

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \\ \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = u \end{cases} \tag{14}$$

obtaining the dynamics

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ u \in <-\alpha, \beta> \end{cases} \tag{15}$$

with initial conditions

$$\begin{cases} x_2(0) = a_1 \\ x_2(0) = a_2 \end{cases} \tag{16}$$

and terminal condition

$$\begin{cases} \dot{x}_1(T) = 0 \\ x_2(T) = 0 \end{cases} \tag{17}$$

Application of Maximum Principle

Suppose $\hat{u}(t), t \in [0, \hat{T}]$ is the optimal solution and $\hat{x}(t)$ is the corresponding optimal trajectory. Further consider the Hamiltonian function

$$H = \psi_1 f_1 + \ldots + \psi_n = \psi_1 x_2 + \psi_2 u \tag{18}$$

From the Hamiltonian function in equation (18), the maximum condition has the form

$$\begin{cases} Max\left(\psi_1 \hat{x}_2 + \psi_2 u\right) = \psi_1 \hat{x}_2 + \psi_2 \hat{u} \quad \text{on } \left[0, \hat{T}\right] \\ u \in <-\alpha, \beta> \end{cases} \tag{19}$$

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Eliminating all terms that do not depend on u reduces the maximum condition in equation (19) to

$$\begin{cases} Max\left(\psi_2 u\right) = \psi_2 \hat{u} \quad \text{on } \left[0, \hat{T}\right], \\ u \in < -\alpha, \beta > \end{cases} \tag{20}$$

Then, we have

$$\hat{u}(t) = \begin{cases} \beta, & \psi_2(t) > 0 \\ ?, & \psi_2(t) = 0 \\ -\alpha, & \psi_2(t) < 0 \end{cases} \tag{21}$$

The adjoint system is obtained as follows:

$$\begin{cases} \overset{g}{\psi}_1 = -\dfrac{\partial H}{\partial x_1} = 0 \\ \overset{g}{\psi}_1 = -\dfrac{\partial H}{\partial x_1} = -\psi_1 \end{cases} \tag{22}$$

with solution

$$\begin{cases} \psi_2 = c_1 \\ \psi_2 = -c_1 t + c_2 \end{cases} \tag{23}$$

Considering the sign of ψ_2 , there are five possible cases. ψ_2 is a linear function and has at most one zero root. We show that ψ_2 cannot be identically zero. Suppose on the contrary that $\psi_2 \equiv 0$, then $c_1 = c_2 = 0$ which implies that $\psi_1 = 0$ which further implies that $\psi = (0,0)$. However, the vector of Lagrange multipliers for the optimal time control problem cannot be zero according to the maximum principle. Hence our assumption that $\psi_2 = 0$ is a contradiction because $\psi \neq (0,0)$.

Now we have four variants of the form of the optimal control $\hat{u}$:

Case 1: $\hat{u}_1(t) = \beta$ on $\left[0, \hat{T}\right]$

Case 2: $\hat{u}_2(t) = -\alpha$ on $\left[0, \hat{T}\right]$

Case 3: $\hat{u}_3(t) = \begin{cases} \beta; & t \in < 0, t_s) \\ -\alpha; & t \in < t_s, T > \end{cases}$

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Case 4: $\hat{u}_4(t) = \begin{cases} -\alpha; & t \in <0, t_s) \\ \beta; & t \in <t_s, T> \end{cases}$

If the $\psi_2(t) = 0$ is satisfied only in isolated time points, then these points correspond to switching time, where the value of optimal control is discontinuously changing.

From case 1, we find all real couples $(a_1.a_2) \in \mathbb{R}^2$ such that $\hat{u}_1$ is the optimal control. Substituting $\hat{u}_1(t)$ into the system yields the general solution,

$$\begin{cases} \dot{x}_1 = x_2 \rightarrow x_1(t) = \frac{1}{2}\beta t^2 + c_2 t + c_1 \\ \dot{x}_2 = \beta \rightarrow x_2(t) = \beta t + c_2 \end{cases} \tag{24}$$

Applying the initial conditions in equation (16) gives the particular solutions

$$\begin{cases} x_1(t) = \frac{1}{2}\beta t^2 + a_2 t + a_1 \\ x_2(t) = \beta t + a_2 \end{cases} \tag{25}$$

Similarly, with the terminal conditions in equation (17), it yields

$$\begin{cases} x_1(T) = \frac{1}{2}\beta T^2 + a_2 T + a_1 = 0 \\ x_2(T) = \beta T + a_2 = 0 \end{cases} \tag{26}$$

Equations (25) and (26) represent a curve given parametrically in the planar space (a_1, a_2). $T > 0$ is a parameter of the curve; if we eliminate this parameter, we get an implicit representation of the curve. Thus,

$$\begin{cases} a_1 = a_1(T) \\ a_2 = a_2(T) \end{cases} \tag{27}$$

From Equation (26)b,

$$\beta T + a_2 T = 0$$

$$T = \frac{-a_2(T)}{\beta}$$

$$T = \frac{-a_2}{\beta} \tag{28}$$

Also from Equations (26)a and (28),

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$$x_1(T) = \frac{1}{2}\beta T^2 + a_2 T + a_1 \qquad = 0$$

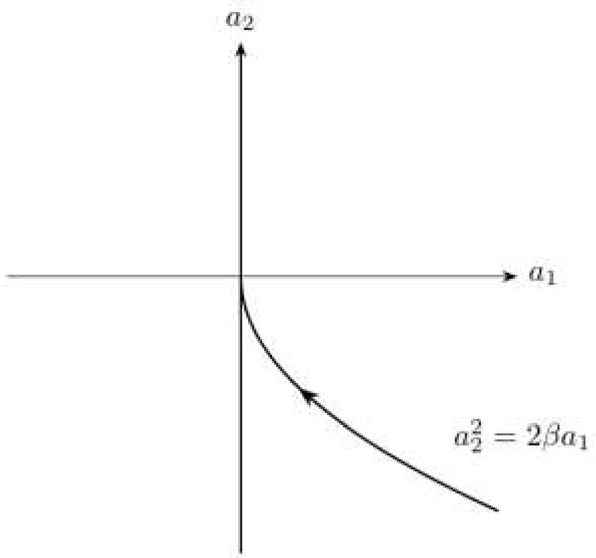
$$= \frac{a_2^{\,2}(T)}{2\beta} - \frac{a_2^{\,2}}{\beta} + a_1(T) = 0$$

Hence,

$$a_2^{\,2} = 2\beta a_1 \qquad (29)$$

The equation (29) is an equation of a parabola. $\hat{u}_1$ is the optimal control if we take the parameter from this part of the parabola for $T > 0$. This is illustrated graphically in Figure 1.

Figure 1: Optimal Control Path of $\hat{u}_1$



From case 2, we find all real couples $(a_1.a_2) \in \mathbb{R}^2$ such that $\hat{u}_2$ is the optimal control. Substituting $\hat{u}_2(t)$ into the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\alpha \end{cases} \qquad (30)$$

gives the general solution

$$\begin{cases} \dot{x}_1 = x_2 \rightarrow x_1(t) = -\frac{1}{2}\alpha t^2 + c_2 t + c_1 \\ \dot{x}_2 = -\alpha \rightarrow x_2(t) = -\alpha t + c_2 \end{cases} \qquad (31)$$

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Note that the system in the equation (30) is the same as the original control system (15) except for that $u = -\alpha$.

Applying initial conditions;

$$\begin{cases} x_1(t) = -\frac{1}{2}\alpha t^2 + a_2 t + a_1 \\ x_2(t) = -\alpha t + a_2 \end{cases} \tag{32}$$

Also, the terminal conditions gives

$$\begin{cases} x_1(T) = -\frac{1}{2}\alpha T^2 + a_2 T + a_1 = 0 \\ x_2(T) = -\alpha T + a_2 = 0 \end{cases} \tag{33}$$

From Equation (33)a;

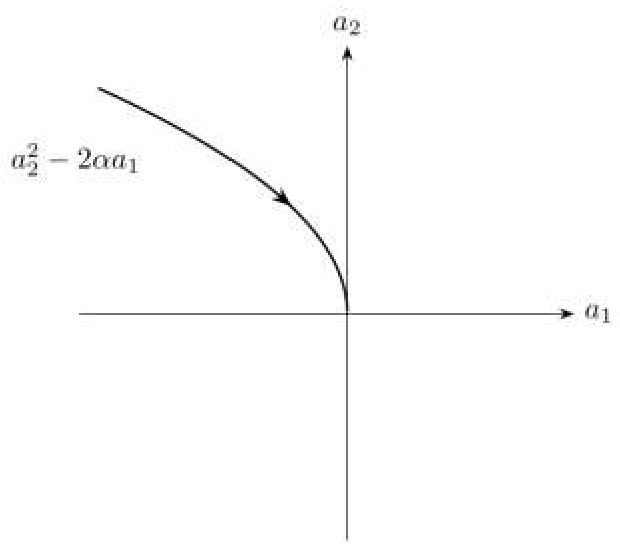
$$T = \frac{a_2}{\alpha} \tag{34}$$

Substituting equation (34) into Equation (33)b, we obtain

$$a_2^{\,2} = -2\alpha a_1 \tag{35}$$

$\hat{u}_2$ is the optimal control if we take the parameter from this part of the parabola displayed in Figure 2.

Figure 2: Optimal Control Path of $\hat{u}_2$



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For case 3, the system to consider is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \hat{u}_3 = \beta \\ t \in <0, t_s> \end{cases}$$

with the particular solution

$$\begin{cases} x_1(t_s) = \frac{1}{2}\beta t_s^{\ 2} + a_2 t_s + a_1 \\ x_2(t_s) = \beta t_s + a_2 \end{cases} \tag{36}$$

Finding $\hat{u}_3$ in the interval $<0, t_s>$ is analogous to the case $\hat{u}_1$. However, we substitute the right endpoint of the time domain $<0, t_s)$ into the general solution, and because of continuity of x_1 and x_2 we obtain the initial conditions for the next time domain $<t_s, T>$.

Thus, in interval $<t_s, T>$, $\hat{u}_3$ is analogous to the case $\hat{u}_2$.

$$\begin{cases} \dot{x}_1 = x_2 \to x_1(t) = -\frac{1}{2}\alpha t^2 + c_2 t + c_1 \\ \dot{x}_2 = -\alpha \to x_2(t) = -\alpha t + c_2 \\ t \in <t_s, T> \end{cases} \tag{37}$$

with initial conditions given by the equation (36).

Applying these initial conditions to Equation (37) gives

$$\begin{cases} x_1(t_s) = -\frac{1}{2}\alpha t_s^{\ 2} + c_2 t_s + c_1 = \frac{\beta t_s^{\ 2}}{2} + a_2 t_s + a_1 \\ x_2(t_s) = -\alpha t_s + c_2 = \beta t_s + a_2 \end{cases} \tag{38}$$

From Equation (38)

$$\begin{aligned} c_2 &= (\alpha + \beta) t_s + a_2 \\ c_1 &= -\frac{1}{2}(\alpha + \beta) t_s^2 + a_1 \end{aligned} \tag{39}$$

and solution

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$$x_1(t) = -\frac{1}{2}\alpha t^2 + \left((\alpha+\beta)t_s + a_2\right)t + a_1 - \frac{1}{2}(\alpha+\beta)t_s^{\ 2}$$
$$x_2(t) = -\alpha t + a_2 + (\alpha+\beta)t_s \tag{40}$$

It is now possible to substitute the terminal condition $x_1(T) = x_2(T) = 0$ since we are in the second domain:

$$\begin{cases} x_1(T) = 0 = -\frac{1}{2}\alpha t^2 + \left((\alpha+\beta)t_s + a_2\right)t + a_1 - \frac{1}{2}(\alpha+\beta)t_s^{\ 2} \\ x_2(T) = 0 = -\alpha t + a_2 + (\alpha+\beta)t_s \end{cases} \tag{41}$$

We find explicit expressions of $t_s = t_s(a_1, a_2)$ and $T = T(a_1, a_2)$ from the equation (41) with the conditions, $0 < t_s < T$. From Equation (41)b,

$$t_s = \frac{\alpha T - a_2}{\alpha + \beta} \tag{42}$$

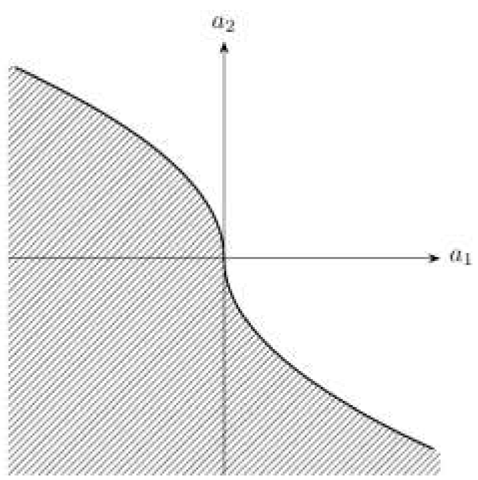
Substituting Equation (42) into Equation (41)a and further calculations yields;

$$T_{1,2} = \frac{-2\alpha a_2 \pm \sqrt{4\alpha a^2 - 4\alpha\beta\left(\alpha a_1(\alpha+\beta) - a_2^{\ 2}\right)}}{2\alpha\beta} \tag{43}$$

The following conditions are consequences of the equation (43). It has a positive discriminant and also T_1, T_2 must be positive.

Analysis of these conditions implies the region of all initial states (a_1, a_2) for which the control $\hat{u}_3$ is optimal is displayed in Figure 3.

Figure 3: Optimal Control of $\hat{u}_3$



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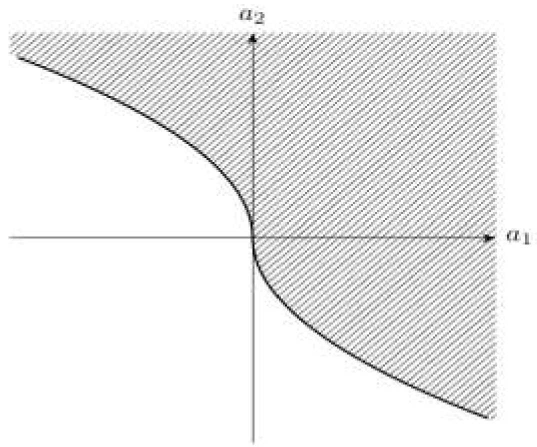
Also, the set of all real couples (a_1, a_2) such that $\hat{u}_4$ is the optimal control is depicted in Figure 4.

Figure 4: Optimal Control of $\hat{u}_4$



An alternative to the above analytical approach is the geometric method. Therefore, we also present a geometrical method that consists of the reconstruction of an optimal trajectory from single pieces corresponding to both values of optimal control $u = \beta, u = -\alpha$.

First, we draw the system of optimal trajectories corresponding to both possible values of the optimal control.

For the first case,

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \beta \\ \hat{u} = \beta \end{cases} \tag{44}$$

has the general solution

$$\begin{cases} x_1 = \dfrac{1}{2}\beta t^2 + c_2 t + c_1 \\ x_2 = \beta t + c_2 \quad ; t > 0 \end{cases} \tag{45}$$

From (45)b,

$$t = \frac{x_2 - c_2}{\beta} \tag{46}$$

Substituting the equation (46) into (45)a, we obtain

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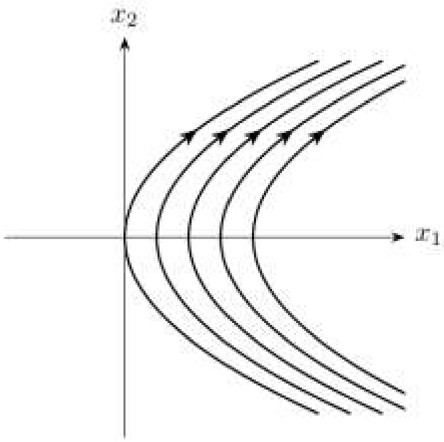
$$x_1 = \frac{1}{2}\frac{(x_2-c_2)^2}{\beta} + \frac{c_2}{\beta}(x-c_2)+c_1 \\ = \frac{x_2^{\ 2}}{2\beta}+C \tag{47}$$

where $C = -c_2^{\ 2}/2\beta + c_1$. A graphical representation of x_1 in the equation (47) is depicted in Figure 5.

Figure 5: Optimal Path of $\hat{u} = \beta$

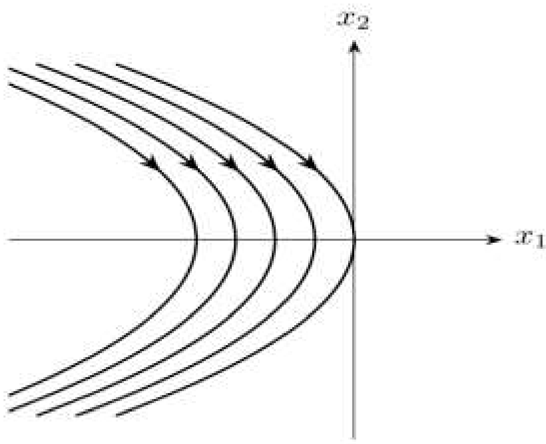


Apply the same to case 2, we obtain

$$x_1 = -\frac{x_2^{\ 2}}{2\alpha} + K \tag{48}$$

where $K = c_1 + c_2^{\ 2}/2\alpha$. A graphical illustration of x_1 in the equation (48) is displayed in Figure 6.

Figure 6: The Optimal Path of $\hat{u} = -\alpha$



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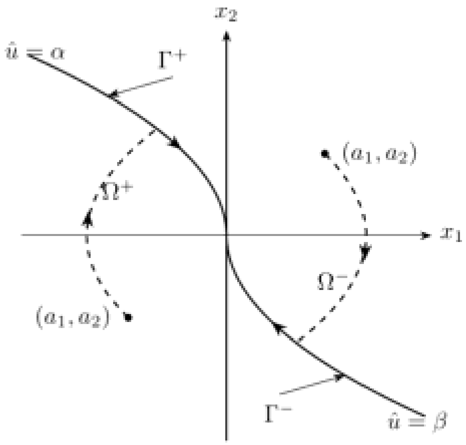
Following the above analysis, cases 3 and 4 can be redescribed as follows: Let (a_1, a_2) be an arbitrary initial state. We find a strategy to convert it to the terminal state $(0,0)$ along the trajectories obtained above, subject to the restriction that at most one switch between both trajectories is allowed. This idea is depicted in Figure 7.

Figure 7: The Switching Cure of $\hat{u}$



For the initial positions above the curve, we are first controlled by $-\alpha$ and after reaching the curve, we switch to β. For the initial position under the curve, we do the opposite, thus first β followed by $-\alpha$.

Let $\Gamma = \Gamma^+ \cup \Gamma^-$ be the *switching curve.* Denote Ω^+ as the set of all couples (x_1, x_2) lying below the switching curve Γ, and above Γ we have Ω^-. Using these notations, we get a simple formula for the optimal control, namely

$$\hat{u} = \begin{cases} \beta; & (x_1, x_2) \in \Omega^+ \cup \Gamma^+ \\ -\alpha; & (x_1, x_2) \in \Omega^- \cup \Gamma^- \end{cases} \qquad (49)$$

The advantage of this expression consists in its ability to react promptly to possible sudden changes in the position or velocity of a controlled object. The disadvantage is that we cannot calculate the switching time t_s and terminal time T, they must be calculated analytically.

CONCLUSION

This study formulated and solved the time-optimal control of a uniform linear motion system using the Pontryagin Maximum Principle. The necessary conditions for optimality confirmed that the optimal control switches between the extreme values $u = -\alpha$ and $u = \beta$, with at most one switch occurring along any optimal trajectory. Phase-plane analysis yielded explicit parabolic switching curves that partition the state space into two regions, each dictating a distinct switching sequence depending on the initial state. The resulting control responds

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promptly to changes in position and velocity, making it suitable for real-time implementation. Future work could extend this framework to nonlinear and multi-dimensional systems, incorporate stochastic disturbances, and explore numerical methods for computing switching and terminal times in more complex optimal control problems arising in robotics, autonomous navigation, and engineering systems.

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