African Journal of Mathematics and Statistics Studies
Volume 2, Issue 1, 2019 (pp. 10-20)
www.abjournals.org

ON THE REPRESENTATIONS OF FINITE 2 – GROUPS WITH MAXIMUM CENTRE

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ABSTRACT: *Jelten B. Naphtali has achieved the minimum number of irreducible representations of finite non abelian groups using the centre. Also, Naphtali et al have determined the representation of finite 2 – groups with minimum centre using the degree equation. In this paper we determine the irreducible representations of finite p – groups with maximum centre where p is prime. We consider this for finite abelian and finite non abelian groups both of prime power order. That is groups of order p^n for $1 \leq n \leq 6$, where n is an integer and p is fixed at p = 2. In this work we employ the class equation as the major theorem in proving our results.*

KEYWORDS: Centre, Abelian, Groupoid, Class Equation, Conjugacy, Finite 2

1. INTRODUCTION

Here we define some basic and fundamental concepts and theorems. We prove some of the theorems to lay a foundation for our results.

1.1 Definition

A groupoid G is a set which is closed for a binary operation in G. That is for which given any $a, b \in G$ of elements of G we have $ab \in G$. If in addition to being closed the set G is also associative then it's called a semi group. That is for which $a(bc) = (ab)c$ where a b and c are elements of G. A semi group in which there exists an element $e \in$ G called the identity element in G such that: $ae = a = ea$ is said to be a monoid. Furthermore, if the monoid possesses an element $a^{-1} \in G$ called the inverse of a such that: $aa^{-1} = e = a^{-1}a$ then it is called a group.

1.2 Definition

A group which possesses the property that for any pair of elements x, y in G we have $xy = yx$ is called an abelian group named after Abel Neils a Norwagean Mathematician. While the abelian property does not hold throughout in a group, whenever this is the case there exist at least a pair of elements say x and y in the group such that $xy \neq yx$. Such a group is said to be non-abelian.

Consequently, the property which any pair of elements are endowed with whenever they are combined by a binary operation determines the abellianess or otherwise of the group.

1.3 Definition

The number of elements in a group G written $|G|$ is called the cardinality of the group. If G is finite of order n we write $|G| = n$ otherwise, $|G| = \infty$ if G has infinite order.

The least number n if it exists such that $a^n = 1$for a in G is called the order of a and we write $o(a) = n$ and $o(a) = \infty$ if no such n exists. An element of order two is said to be an involution that is $o(a) = 2$. In this case $a = a^{-1}$ so a and a^{-1} have the same order.

The identity element is the only element of order 1. However, if $a, b \in G$ are such that $a = g^{-1}bg$ for some g in G then a and b are conjugate elements with $o(a) = o(b)$.

1.4 Definition

A non-empty subset N of a group G is said to be a subgroup of G written $N \leq G$, if N is a group under the operation inherited from G. If $N \neq G$, Then N is a proper subgroup of G. A subgroup N of G such that every left coset is a right coset and vice versa is called a normal subgroup of G. That is $Nx = xN$ or $x^{-1}Nx \leq N$ and we write $N \triangleleft G$. In an abelian group every subgroup is normal. We call G a simple group if the only normal subgroups of G are the identity element and the group G itself.

1.5 Definition

Let G be a group and $H < G$. For $q \in G$ the subset Hq of G is called the right coset of H defined as Hq = $\{hq{:}h \in H\}$. Distinct right cosets of H in G form a partition of G. That is, every element of G is precisely in one of them. Left coset is similarly defined. If G is abelian, we talk of cosets of H. The number of distinct right cosets of H in G is written as $|G{:}H|$ and is called the index of H in G. If G is finite so is H and G is partitioned into $|G{:}H|$ cosets each of order $|H|$. It follows that: $|G{:}H| = \frac{|G|}{|H|}$.

The next theorem gives a relationship between the order of a group and the order of its subgroups.

1.6 Theorem

If a group G is finite and H is a subgroup of G then the order of H divides the order of G.

Proof

By definition 1.5 we have that the right cosets of H form a partition of G. Thus, each element of G belongs to at least one right coset of H in G and no element can belong to two distinct right cosets of H in G. Therefore, every element of G belongs to exactly one right coset of H. Moreover, each right coset of H in G contains $|H|$ elements. Therefore, if the number of right cosets of H in G is n, then $|G| = n|H|$. Hence the order of H divides the order of G.

Note that H and G/H are subgroups of G from theorem 1.8 $|H|$ and $|G{:}H|$ divide $|G|$. A consequence of this is that if H is a subgroup of G then, $|G| = |G{:}H||H|$.

1.7 Definition

If $g \in$ G is such that g generates the group G we say G is cyclic written as $G = < g >$. That is $G = \{ g^n : n \in Z\}$ if G is finite. In the infinite case, G is cyclic if the powers of g exhaust G. Here, every element $b \in G$ can be expressed as $b = g^m$ for some integer m. In this sense we say $g \in G$ generates G.

An important fact about cyclic groups is that they are generally abelian. Hence the next corollary.

1.8 Corollary

(i) If G is a group such that $g^2 = 1$ for all g in G then G is abelian.

(ii) $g^n = 1$ if and only if $(x^{-1}gx)^n = 1$

From definition 1.5 and theorem 1.6 we have that:

1.9 Corollary

If G is a finite group and q is in G then the order of q is a divisor of the order of G since $<q>$ is a subgroup of G generated by q. Furthermore, the order of an element of G divides $|G|$. In particular $a^{|G|} = 1$ where G is finite

Next, we define an important concept in this paper.

1.10 Definition

Let a, $q \in G$. Then a is conjugate to q in G if there exist an element $g \in$ G such that $q = g^{-1}a\, g$. The set of all elements of G that are conjugate to a in G is called the conjugacy class of a in G which we denote by $C(a)$ and defined as: $C(a) = \{ g^{-1}\, a\, g |\, g \in G\}$ The identity element belongs to its own class. We note that $C(a)$ is a subgroup of G and by theorem 1.6 its order divides that of G. Subgroups belonging to the same conjugacy class are conjugates. However conjugate elements lie in the same conjugacy class and have the same order. If the group is abelian then, $C(a) = \{a\}$ for all a in G. This shows that the concept of conjugacy is trivial in the abelian case.

1.11 Definition

The centre $Z(G)$ of a group G is the set of all elements z in G that commute with every element q in G. We write $Z(G) = \{z \in G|\ zq = qz$, for all $q \in G\}$. The centre of a group is the union of conjugacy classes of size one.

$Z(G)$ is a abelian normal subgroup of G. For abelian and non-abelian groups the minimum size of the centre is 1and 2 respectively. The centre of a group G is its subgroup of largest order that commute with every element in the group.

1.12 Definition

The centralizer C_G(q) of an element q in G is the set of all elements g $\in$ G that commute with q. This is defined as: $C_G(q) = \{g \in G : gq = qg$, for some $q \in G\}$.

1.13 Theorem

All elements of a conjugacy class have the same order. The converse of this is false. That is elements of the same order in a group are not always conjugates. The following remark shows a very strong relationship between the conjugacy class of an element and its index in the group

1.14 Remark

The centralizer is a subgroup of G but not a normal subgroup in general and hence its order divides |G|. The index of $C_G(q)$ in G is the size of the conjugacy class C(q) of q in G. That is

$|C(q)|=|G:C_G(q)|$. If $q \in Z(G)$ then $|C(q)|=1$ and $q^{-1}gq=q$. So that $C_G(q) = G$. Consequently, the quotient of G by $C_G(q)$ is not a group

As a consequence of theorem 1.6 the size of the conjugacy class is a factor of the order of *G*. If we choose a single representative element x_i from each conjugacy class, then by the fact that the conjugacy classes are disjoint we have that: $|G| = \sum|G:C_G(x_i)|$. This gives rise to an important theorem. The class equation which we state in 1.24 (iii)

The definition that follows defines a key concept.

1.15 Definition

A *p* - group is a group whose order is a power of a prime *p*. If a group *G* has order p^m where *p* is a prime and *m* is a positive integer, then *G* is a *p* - group. Consequently 2- groups are p – groups with p = 2.

We collect some properties of *p* - groups in the following lemma.

1.16 Lemma

Let *G* be a group of order p^n, with $n \geq 1$ and $H \leq G$ if

(1) $\{1\} \neq H \lhd G$, then $H \cap Z(G) \neq \{1\}$. In particular $Z(G) \neq \{1\}$;

(2) $K \leq Z(G)$ and G/K is cyclic then *G* is abelian

(3) $n \leq 2$, then G is abelian

1.17 Definition

The subgroup $G^{/}$ or $[G,G]$ of a group *G* generated by the elements of the form $sts^{-1}t^{-1}$, for all $s, t \in G$ is called the derived group or commutator subgroup of *G*. We write $[s,t] = sts^{-1}t^{-1}$ and call this the commutator of *s* and *t*. Thus: $G^{/} = \{ [s,t] : s, t \in G \}$.

The derived group is normal in *G*. It is the largest abelian quotient of *G* hence we state the next theorem without prove.

1.18 Theorem

The derived subgroup $G^{/}$ of a finite group *G* is the unique minimal normal subgroup of *G* such that $G/G^{/}$ is abelian. That is if G/N is abelian then $G^{/} \leq N$ and $G/G^{/}$ is abelian.

The next corollary outlines properties of conjugacy class proved by [3]

1.19 Corollary

If G is a finite group, then:

(i) every group is a union of its conjugacy classes and distinct conjugacy classes are disjoint;

(ii) conjugacy class is an equivalence relation where the equivalence classes are the conjugacy classes.

A relationship between the centre of G and the centralizer of the elements of G is given by:

1.20 Lemma

The centre Z(G) of a group G is the intersection of the centralizers $C_G(a)$ of elements *a* in G.

From [2] and definition 1.10 we have a theorem on the relationship between the order of the conjugacy class of an element of *G* and its index in G

1.21 Theorem

Let G be a finite group and $q \in G$, then the conjugacy class C(q) of q in G is given by:

$|C(q)| = |G:C_G(q)| = \dfrac{|G|}{|C_G(q)|}$. The sum of the centralizers of all elements of *G* can be separated into sums of the centralizers of all the elements from each conjugacy class of *G*.

Next from 1.19 we relate normal subgroup to conjugacy class as:

1.22 Proposition

Let $N \leq G$, then $N \lhd G$ if and only if *N* is the union of conjugacy classes of *G*.

[6] proves the following theorem.

1.23 Theorem

If a finite group *G* has a centre *Z*(*G*) and *G*/*Z*(*G*) is cyclic then *G* is abelian.

1.24 Theorem

Let *G* be a finite group then from remark 1.14

$|G| = \sum|G: C_G(q_i)|$ .... (*i*),

where the sum runs over the elements from each conjugacy class of *G*. \

From Theorem 1.21, equation (i) becomes

$$|G| = |Z(G)| + \sum|G: C_G(q_i)| \quad \text{...} \quad (ii)$$

Here the sum in (*ii*) runs over q_i from each conjugacy class such that q_i is not an element of $Z(G)$. From theorem 1.21 and equation (*ii*) above we have:

$$|G| = |Z(G)| + \sum|C(q_i)| \qquad \text{(iii)}$$

1.25 Remark

In the abelian environment, the sum in equation (*iii*) of Theorem 1.24 vanishes. Consequently, the class equation is relevant only when we are in the non-abelian environment. The fact that each element of $Z(G)$ forms a conjugacy class containing just itself gives rise to the class equation. From Theorems 1.6 and 1.24 we have:

1.26 Proposition

If the order of a finite group G is a power of a prime p then G has a non-trivial centre. Equivalently the centre of a p - group contains more than one element.

Proof

Let G be the union between its centre and the conjugacy classes say J_i of size greater than 1.

Then from equation (*iii*) of Theorem 1.24, $|G| = |Z(G)|+\sum|C(J_i)|$

Each conjugacy class J_i has size of a power w say of prime p such that $w \geq 1$. In this case $w = 0$ for the conjugacy classes whose elements are central elements. Since each conjugacy class J_i has size a power of p then $|J_i|$ is divisible by p. Furthermore, as p divides $|G|$, it follows that p also divides $|Z(G)|$. Accordingly, $Z(G)$ is non- trivial.

Proposition 1.26 implies that there are elements of G other than the identity that commute with every element of G.

From [3] and [1] we have:

Theorem 1.27

Every group of order p^2 where p is a prime number is abelian.

Proof

By Proposition 1.26, the centre is non trivial. Thus, by Theorem 1.6 the only remaining possibilities are $|Z(G)| = p$ or p^2. If $|Z(G)| = p^2$, then we are done. If, however $|Z(G)| = p$, then there would exist some a in G such that a is not in the centre of G. For a we have that $Z(G) \subseteq C_G(a)$ and $|Z(G)| \neq |C_G(a)|$because the centralizer of a contains a. That forces $C_G(a) = G$, which means that a is in the centre. This contradicts our hypothesis. The contradiction shows that the case $|Z(G)| = p$ cannot occur. The result follows.

[2] proves the next theorem.

1.28 Theorem

If G is a finite non abelian group, then the maximum possible order of the centre of G is ¼$|G|$.That is, $|Z(G)| \leq 1/4|G|$.

In the next proposition we relate the conjugacy class to the irreducible representation of a finite group G as in [2]

1.29 Proposition

(i) The number of the irreducible representations of any group G is equal to the number of conjugacy classes of G;

(ii) Every irreducible representation of a commutative group G over C, the set of complex numbers is one dimensional.

Consequently, we talk of the the size of the conjugacy class and equivalently the number of the irreducible representations.

The next theorem [4] determine the minimum and maximum number of irreducible representations of prime degree of non-commutative group using its centre. This requires the theorem 1.28

1.30 Theorem

Let G be a finite non abelian group of order r^w such that $|Z(G)| = r^t$ with $t < w$, r a prime number, t and w positive integers. Then G:

(i) does not have an irreducible representation of degree r greater than 1 whenever $t = 0$;

(ii) has its minimum number of irreducible representations of degree r greater than 1 whenever $w = 3$.

From [5] we have the next theorem

1.31 Theorem

Given that a finite group G is of prime power order with centre Z(G), then we count the number of conjugacy classes from the centralizer as follows: $|C| \leq \frac{1}{4}(3|G| - |C_G(x)|)$.

Proof

$$|G| = |Z(G)| + \sum_{i=1+|Z(G)|}^{|C|-|Z(G)|} |G : C_G(x_i)|, \; x \notin Z(G) \quad \text{with } |C(x)| \geq 2.$$

So that $|G| \geq |Z(G)| + 2(|C| - |Z(G)|)$

$$|G| \geq \frac{1}{2}C_G(x) + 2|C| - 2|Z(G)|. \text{ That is } |G| \geq \frac{1}{2}C_G(x) + 2|C| - \frac{1}{2}|G|$$

This leads to $2|G| \geq C_G(x) + 4|C| - |G|$, so $3|G| \geq C_G(x) + 4|C|$

We have $\dfrac{3|G| - |C_G(x)|}{4} \geq |C|$ which reduces to $\frac{1}{4}(3|G| - |C_G(x)|) \geq |C|$

That is $|C| \leq \frac{1}{4}(\,3|G| - |C_G(x)|\,)$

From [6] we have the next two theorems without prove

1.32 Theorem

Let G be a finite p-group with centre |Z(G)|. Then the number of irreducible representations of G is |G| if the group G is abelian with centre at its minimum given $G^{/} \leq G$.

This is a trivial result

1.33 Theorem

Let G be a finite non abeian 2-group with centre |Z(G)| fixed at its minimum. Then the number of irreducible representations of G is given by $|C| \leq \frac{|G|+12}{4}$ given that $G^{/} \leq G$.

2. OUR RESULTS

2.1 Theorem

Given that G is a finite abelian 2 – group. Then the number of irreducible representations of G is |G| given that the centre of *G* has maximum size.

Proof

From the class equation

$|G| = |Z(G)| + \sum_{|Z(|G)|+1}^{|C|} |C(q_i)|$, where $q \in G$ but $q \notin Z(G)$ and |C(q)| ≥ 2.

$$|G| \geq |Z(G)| + 2(\,|C| - |Z(G)|\,)$$

$$|G| \geq |2|C| - |Z(G)|$$

$$|G| + |Z(G)| \geq 2|C|$$

$$\frac{|G| + |Z(G)|}{2} \geq |C|$$

Recall that for an abelian group *Z*(*G*) = *G* hence,

$\frac{|G| + |G|}{2} \geq |C|$ which reduces to:

$$|G| \geq |C|$$

Giving the maximum number of the irreducible number of representations for an abelian group which is finite.

Theorem 2.2

For a finite non - abelian 2 – group with a maximum centre the maximum number of irreducible representations is at most $\frac{|5}{8}|G|$.

Proof

From the class equation we have that

$$|G| = |Z(G)| + \sum_{|Z(|G)||+1}^{|C|} |C(q_i)|$$, where $q \in G$ but $q \notin Z(G)$ and $|C(q)| \geq 2$

So that

$$|G| \geq |Z(G)| + 2(|C| - |Z(G)|)$$

$$|G| \geq |2|C| - |Z(G)|$$

$$|G| + |Z(G)| \geq 2|C|$$

$$\frac{|G| + |Z(G)|}{2} \geq |C|$$

From theorem 1.28, for finite non abelian groups we have that max $|Z(G)| \leq 1/4|G|$.

Hence

$$\frac{|G| + \frac{|G|}{4}}{2} \geq |C|$$

Simplifying the left-hand side of the last inequality we get

$\frac{5}{8}|G| \geq |C|$, hence the result.

DISCUSSION

We consider the result in theorem 2.1 one as trivial though it plays a fundamental role in the theory of finite groups and the classification of finite groups. Next, we illustrate theorem 2.2 in what follows.

Since the group under consideration is of prime power order, we have that $|G| = 2^n$ with $n > 2$. The first result hold for $1 \le n \le 2$. The cases for $2 < n < 6$ were considered as follows:

For $n = 3$, $|G| = 8$ hence $|C| \le \frac{5}{8}(8) = 5$

For $n = 4$, $|G| = 16$ hence $|C| \le \frac{5}{8}(16) = 10$

For $n = 5$, $|G| = 32$ hence $|C| \le \frac{5}{8}(32) = 20$

These agree with results obtained by [5].

CONCLUSION

The values obtained in the discussion section agree with those of Jelten and apine who counted the maximum number of the conjugacy classes for groups of prime power order with focus on the non-abelian case. An interesting part of this result is that with the centre fixed at its maximum, we obtained the corresponding maximum number of the irreducible representations for the 2 – groups under consideration. Our results are highly simplified with reduced number of variable and so easy to apply.

REFERENCES

[1] B. Houshang. and Hamid, M. (2009). A note on p-groups of order $\le p^4$. Proc. Indian Acad. Sci. (Math. Sci.), **(2009),** vol 2, 137-143.

[2] C. Cody, (2010) Commutativity in non abelian groups. www.whitman.edu/mathematics/.../2010/seniorproject.codyclifton, (2010) 1-18. Retrieved on 2/9/ 2012.

[3] J. Gordon and Martin, L.(2001) Representation and characters of groups (second edition). Cambridge University press.

[4] J. B. Naphtali,(2016) The irreducible components of the representations of finite non abelian groups. Unpublished Ph. D thesis, Mathematics Department, University of Jos. Nigeria.

[5] J. B. Naphtali and E, Apine (2015) Counting the conjugacy classes of finite groups from the Centralizer. International journal of mathematics and statistics invention, vol. 3 no. 1, pages 38 – 47.

[6] Jelten et al (2017) On The Representations of Finite 2 – Groups With Minimum Centre. International Journal of Innovative Mathematics, Statistics and Energy Policies, vol. 5(3) July to September; pages 1 – 14.
[7] R. Mark, (2011). Notes on group theory. https://www2,bc.edu/-/development-group-Theoryhtml, p. 4-9. Retrieved on 9/9/ 2012.
[8] S. Louis, (1975) Introduction to abstract algebra. New York: McGraw – Hill Inc.
[9] W. Laderman and Weir, A J (1966) Introduction to group theory, (2nd edition). Addison Wesley Longman Inc. New York.