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ON THE PROBABILITY OF GAMBLER'S RUIN SYSTEM

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ABSTRACT: *A typical Gambler's ruin problem was considered in this work by applying theory of difference equations. The expected gains and duration of the game were generated for both fair and unfair games. In order to develop the probabilities that a Gambler meets the target of initial capital units, the frequency on steps hitting zero or initial capital was computed. As a particular unit tends to infinity, the sum of probabilities of ruin and success was established to be unity. Boundary conditions were set up to develop the generating function for the duration of the game. To ascertain these, the conditions were validated by finding the generating functions for probability of gambler's ruin at zero and unit absorptions. The illustrated example showed that the expected gain is 0 when the game is fair and negative when it is unfair. While the expected duration of the game was found to be infinite for a fair game, an infinitely prolonged game series is infeasible.*

KEYWORDS: Gambler's Ruin, Difference Equations, Expected Game Duration, Expected Game Gains, Generating Functions.

INTRODUCTION

A classical gambling ruin problem is a one-dimensional random walk with reflecting and elastic barriers, proposed by Pascal and Fermat 1656. Its applications abound in different areas of human life [14];[11];[10];[13];[6];[7][8] among others.

When the assumption that a random walk is symmetric is violated, the sum of the non-negative real numbers p and q is 1, and the common function of the jumps of the random walk is

$$f(x)=\begin{cases} p, if \ \ x=1 \\ \\ q, if \ \ x=-1 \end{cases}$$

The random walk represents a sequence of tosses of a weighted coin, with a head appearing with probability. An alternative formulation of this scenario is that of a gambler playing a sequence of games against an opponent where, in each game, the gambler has a probability p of winning. This random walk formulation, results into a gambler's ruin problem, which has been principally established [5];[9];[3];[12].

Given a gambler who wins or loses a unit of ₦1 with probabilities p(0 < p) and q(q < 1) respectively. Suppose his initial capital is ₦m and he plays against an adversary with initial capital ₦(a-m) so that the sum of the capital is ₦a. The game continues until one of the

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players is ruined (the gambler's capital either reduces to zero or increases to ₦a. In this regard, it is assumed that the gambler plays against an infinitely rich adversary, and there is only one absorbing state since the gambler's stake is 0 [4].

The gambler's ruin problem can also be seen from the perspective of Markov chain. Every state is a reflection of the amount of investment the gambler at any time. The gambler wins or loses ₦a while moving from state a to either left or right. The game terminates when the gambler loses all of his investments or gets at his goal of winning ₦a. The probability of ruin is the likelihood of getting to a state 0 before state a [1].

In this work, the probability of gambler's ruin using difference equations is generated along with gambler's expected game and duration. The generating function for the game duration is also developed.

METHODOLOGY

Generating Probability of Gambler's Ruin using Difference Equation

A sequence in which every term is a function of its predecessors, so that

$$x_{r+k} = f(x_{r,}\ x_{r+1,}\ \ldots x_{r+k-1}),\ r \geq 0 \qquad (1)$$

is known to have satisfied the recurrence relation (1). When f is linear, equation (1) is termed a difference equation of order k:

$$x_{r+k} = a_0 x_r + a_1 x_{r+1} + \ldots a_{k-1} x_{r+k-1} + g(r), \quad a_0 \neq 0 \ldots\ldots\ldots (2)$$

If g (r) = 0, then equation (2) becomes homogeneous equation:

$$x_{r+k} = a_0 x_o + a_1 x_{r+1} + \ldots a_{k-1} x_{r+k-1},\ a_0 \neq 0 \qquad (3)$$

Suppose $E^r(q) = q_{m+r}$ and $\Phi(E)q_m = f(m)$, where E is a shift operator and Φ(E) is a polynomial in E. If f(m) =0, then equation (2) is homogeneous and $\Phi(E)q_m = 0$

Given that $\Phi(E) = (E-\alpha_1)(E-\alpha_2),\ldots(E-\alpha_r),\ \alpha_i + \alpha_j, \forall\, i = j,$ the general solution of $\Phi(E)q_m = 0$ is $q_m = a_1\alpha_1^m + a_2\alpha_2^m + \ldots + a_m\alpha_r^m$, where $a_1, a_2, \ldots a_r$ are constants.

However, if $\alpha_1 = \alpha_2 = \ldots = \alpha_r$, then the general solution of $\Phi(E)q_m = 0$ becomes $q_m = \lambda\alpha^r(a_\alpha + a_{\alpha+1}m + \ldots + a_{\alpha+r-1}m^{r-1})$, where $a_{\alpha's}$ are constants.

If $f(x) \neq 0$, the non-homogeneous equation, $\Phi(E)q_m = f(m)$, is solvable. Thus, the homogeneous solution y_k of $\Phi(E)q_m = 0$, the particular solution of y_p of $\Phi(E)q_m = f(m)$ and the general solution of $\Phi(E)q_m = f(m)$ given as $q_m = y_k + y_p$ are obtainable.

Let $f(m) = HU^m$, where H and U are constants. In order to obtain the particular solution $\Phi(E)q_m = f(m)$, these three cases are considered in the study:

(a) $y_p = \dfrac{Hm(m-1)(m-2)...(m-r+1)U^{m-r}}{\frac{d^r}{dE^r}[\Phi(E)]_U}$, (4)

Equation (4) holds when $U = \alpha_i$ are r-repeated linear factors of $\Phi(E)$

(b) $y_p = \dfrac{HU^m}{\Phi(U)}$ (5)

Equation (5) holds if $U_i = \alpha_i$ *and for any* α_i, $E-\alpha_i$ is a factor $\Phi(E)$

(c) $y_p = \dfrac{HmU^m}{\frac{d}{dE}[\Phi(E)]_U}$ (6)

Equation (6) also holds *if* $U = \alpha_i$ and $E-\alpha_i$ are non-repeated roots of $\Phi(E)$.

Probability of Gambling Ruin and Success

Let q_m and p_m denote the probabilities that a gambler gets ruined and wins a game respectively. Thus, p_m = p(random walk hits a before it hits 0 / starts from m) and q_m = p(random walk hits 0 before it hits a / starts from m). At the end of the first attempt, the gambler's investment becomes ₦(m+1) given that the attempt resulted in failure. The gambler eventually gets ruined from m stakes.

Given that the difference equation:

$$q_z = pq_{m+1} + qq_{m-1}, \quad 1 < m < a-1 \qquad (7)$$

The probability that the gambler gets ruined can be established when m = 0 and q_o =1. Also, the probability that the gambler wins is also asserted when m = a and q_a = 0 [2].

Therefore, with the initial conditions q_o=1 and q_a = 0, and applying the shift operator, q_m satisfies equation (7):

$$\begin{aligned} E(q_m) &= pE(q_m) + qE^{-1}(q_m) \\ &= q_z(pE + qE^{-1}) \\ &=> (pE + qE^{-1} - E^0)q_m = 0 \\ &=> pE^2 + q - E = 0 \\ &=> E^2 - \frac{1}{p}E + \frac{q}{p} = 0 \\ &=> (E-1)(E - \frac{q}{p}) = 0 \\ &=> \alpha_1 = 1 \quad and \quad \alpha_2 = \frac{q}{p} \qquad (8) \end{aligned}$$

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The general solution is expressed as

$$\begin{aligned} q_z &= A\alpha_1^m + B\alpha_2^m \\ &= A + B\left(\frac{q}{p}\right)^m, \; p \neq q \end{aligned} \tag{9}$$

If p = q, then equation (8) becomes

$$(E-1)(E-1) = 0 \tag{10}$$

With the repeated roots, $q_m = A + Bm$

The general solution of equation (8) is given as:

$$q_m = \begin{cases} A + B\left(\frac{q}{p}\right)^m, & p \neq q \\ A + Bm, & p = q \end{cases} \tag{11}$$

When $p \neq q$, and with the initial condition $q_o = 1$ and $q_a = 0$

$$q_o = A + B, \quad q_a = A + B\left(\frac{q}{p}\right)^a, \quad A = \frac{\left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^a} \quad \textit{and} \quad B = \frac{1}{1-\left(\frac{q}{p}\right)^a}$$

The probability of gambler's ruin when $p \neq q$ is

$$q_m = \left(\frac{\left(\frac{q}{p}\right)^m - \left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^a}\right) \tag{12}$$

When p = q, $q_o = 1$, and $q_a = 0$, the probability of gambler's ruin is

$$q_z = \frac{a-m}{a} \tag{13}$$

The probability that the gambler wins the game is obtained from equations (12) and (13) by replacing p, q, m by q, p and (a-m) respectively:

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$$p_m = \begin{cases} \left(\frac{p}{q}\right)^{q-m} - \left(\frac{p}{q}\right)^{a}, if\ p \neq q \\ 1 - \left(\frac{a-m}{a}\right), \quad if\ \ p = q \end{cases} \tag{14}$$

Theorem 1:

An infinitely prolonged series of games is infeasible.

Proof:

Showing that $p_m + q_m = 1,$ for both $p = q$ and $p \neq q$ must be established.

The limiting case $a = \infty$ corresponds to a game against an infinitely rich opponent.

Let $a \to \infty$ in equations (12) and (13) to have

$$p_m = \left(\frac{q}{p}\right)^m = 1 \tag{15}$$

$$\begin{aligned} => P(path\ \ hits\quad 0\quad ever) &= P(\bigcup_{a=m+1}^{\infty} \{path\quad hits\quad 0\quad before\quad it\quad hits\quad a\}) \\ &= \lim_{a\to\infty} P(path\quad hits\quad 0\quad before\quad it\quad hits\quad a) \end{aligned} \tag{16}$$

For a gambler playing a game against an infinitely rich adversary, the probability of the gambler's ruin is

$$q_m = \begin{cases} 1, & if\ p \leq q \\ \left(\frac{q}{p}\right)^m, & if\ \ p > q \end{cases} \tag{17}$$

Gambler's Expected Gain

Suppose a gambler with an initial capital ₦m plays an infinitely rich opponent who is always willing to play. The gambler adopts the strategy of stopping the game at his convenience by playing until he either loses or increases his capital by ₦a with a net gain of ₦(a-m).

Let q_m be the probability of the gambler losing and 1- q_m be the probability of the gambler winning. The gambler's ultimate gain or loss is a Bernoulli random variable, G, where G assumes the value ₦m with probability q_m and ₦(a-m) with probability 1- q_m [2].

The expected value is expressed as

$$\begin{aligned} E(G) &= (a-m)(1-q_m)+(-m)q_m \\ &= (1-q_m)a-m \\ &= ap_m - m \\ &= 0 \end{aligned} \tag{18}$$

Equation (18) is true, if the game is fair (p = q = ½). But if the game is fair ($p \neq q$; p <½<q),

$$E(G)=\left(1-\frac{\left(\frac{q}{p}\right)^a-\left(\frac{q}{p}\right)^m}{\left(\frac{q}{p}\right)^a-1}\right)a-m$$

$$=\frac{\left(\frac{q}{p}\right)^m-1}{\left(\frac{q}{p}\right)^a-1}a-m$$

$$=\left(\frac{\left[\left(\frac{q}{p}\right)^m-1\right]a}{\left[\left(\frac{q}{p}\right)^a-1\right]m}-1\right)m \tag{19}$$

The Expected Game Duration

Let the finite expectation of the duration of the game be D_m. If the first attempt results in success, the game continues as if the initial position had been m+1. The conditional expectation of the duration assuming success at first attempt is $D_{m+1}+1$. However, if the first attempt had resulted in failure, the game continues as if the initial capital had been m-1 and the expected duration of the game is D_{m-1}. Thus, the expected duration of the game, D_m satisfies:

$$D_m = pD_{m+1}+qD_{m-1}+1, \quad 0<m<a \tag{20}$$

With the boundary conditions $D_o = 0$, $D_a = 0$ and from equation (20),

$$E^0 D_m = pED_m + qE^{-1}D_m + 1 \tag{21}$$

$$\left(pE+qE^{-1}-E^0\right)D_m = -1 \tag{22}$$

$$\Phi(E)D_m = -1 \tag{23}$$

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The homogeneous part of equation (22) is equal to 0, so that $\Phi(E)=0 \quad and \quad D_m \neq 0$.

$$\begin{aligned}\Phi(E) &= pE + qE^{-1} \\ &= pE^2 + q - E \\ &= E - \frac{1}{p}E + \frac{p}{q} \\ &= (E-1)(E-\frac{q}{p}) = 0 \qquad (24)\end{aligned}$$

$$=> \alpha_1 = 1, \quad \alpha_2 = \frac{q}{p}$$

The solution to the homogeneous part of equation (22) has two cases:

Case I: If $p \neq q$, $D_m = A + B\left(\frac{q}{p}\right)^m$

From equation (23), $\Phi(E)D_m = f(m) \quad so \quad that \quad \Phi(E)D_m = -\frac{1}{p}$

The particular solution y_p of $\Phi(E)q_m = f(x)$ from equation (4) is given as

$$y_p = \frac{HmU^m}{{}^{d}\!/_{dE}[\Phi(E)]_U} = \frac{-\frac{1}{p}m(1)^m}{\left[2E - \frac{1}{p}\right]E=1}$$

$$= \frac{-\frac{m}{p}}{\frac{2p-1}{p}}$$

$$= \frac{m}{q-p} \qquad (25)$$

The general solution denoted by

$$\begin{aligned}D_m &= y_h + y_p \\ &= A + B\left(\frac{q}{p}\right)^m + \frac{m}{q-p} \qquad (26)\end{aligned}$$

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Using the initial conditions, $D_o = 0$ and $D_a = 0$,

$$D_o = A + B + \frac{0}{q-p} = 0 \tag{27}$$

$$D_a = A + B\left(\frac{q}{p}\right)^a + \frac{a}{q-p} - 0 \tag{28}$$

Hence,

$$A = \frac{\dfrac{q}{p-q}}{1-\left(\dfrac{q}{p}\right)^a} \quad , \quad B = -\frac{\dfrac{a}{p-q}}{1-\left(\dfrac{q}{p}\right)^a} \quad and$$

$$D_m = \frac{m}{q-p} - \frac{a}{q-p}\left[\frac{1-\left(\dfrac{q}{p}\right)^m}{1-\left(\dfrac{q}{p}\right)^q}\right], \quad p \neq q \tag{29}$$

Case II: When p = q and there are repeated roots, the solution of the homogeneous part is $D_m = A + B_m$.

Given r = 2 $H = -\frac{1}{p}$, U = 1 with $\Phi(E) = E^2 - \frac{E}{p} + \frac{p}{q}$.

$$\begin{aligned} y_p &= \frac{Hm(m-1)(m-2)\ldots(m-r+1)U^{m-r}}{\dfrac{d^r}{dE^r}[\Phi(E)]} \\ &= \frac{Hm(m-1)(m-2)\ldots(m-2+1)U^{m-2}}{\dfrac{d^2}{dE^2}\left[E^2 - \dfrac{1}{p}E + \dfrac{q}{p}\right]} \\ &= \frac{\left(-\dfrac{1}{p}\right)^m (m-1)(1)^{m-2}}{2} \\ &= m(1-m) \end{aligned} \tag{30}$$

For p = q, $D_o = 0$, $D_a = 0$ and the general solution, $D_a = A + Bm + m - m^2$,

the expected duration of the game is $Dm = m(a-m)$.

Thus, the expected duration of the game is expressed as

$$D_m = \begin{cases} m(a-m), & p=q \\ \dfrac{m}{q-p} - \dfrac{a}{q-p}\left[\dfrac{1-\left(\dfrac{q}{p}\right)^m}{1-\left(\dfrac{q}{p}\right)^q}\right], & p \neq q \end{cases} \tag{31}$$

Table 1: Illustrative Example of Gambler's Ruin Problem

p	q	m	a	p_m	q_m	E(G)	D_m
$\frac{1}{2}$	$\frac{1}{2}$	90	100	0.9000	0.1000	0	900
$\frac{1}{2}$	$\frac{1}{2}$	900	1000	0.9000	0.1000	0	90000
$\frac{1}{2}$	$\frac{1}{2}$	8000	10000	0.8000	0.2000	0	16,000,000
$\frac{9}{20}$	$\frac{11}{20}$	9	10	0.7899	0.2101	-1	11
$\frac{9}{20}$	$\frac{11}{20}$	99	100	0.8182	0.1818	-17	171.8

The expected game gain is constant at zero level and negative when $p=q$ *and* $p \neq q$ respectively.

Finite Time Gambler's Ruin

Let the probability of being in a state c after n-steps / initially at a state d be given by $P_{c,d}^{(n)}$, where c = 0,1,…, m.

If $P^{(n)} = \begin{bmatrix} 1 & 0 & 0 & 0. & . & . & 0 \\ q & 0 & p & 0. & . & . & 0 \\ 0 & q & 0 & p. & . & . & 0 \\ 0 & 0 & 0 & 0. & . & . & 1 \end{bmatrix}$, then $P^{(n)}$ has $P_{c,d}^{(n)}$ entries, where $P_{c,o}^{(n)}$ is the probability that by the nth bet, the Gambler has no investment left and $P_{c,d}^{(n)}$ is the probability that by nth bet, the Gambler meets the target of m units.

In order to find $P_{c,d}^{(n)}$, the number of lattice paths from c to d in n steps not hitting 0 or m must be known.

Let $B_{c,d}^{(n)}(m)$ denote the collection of all lattice paths from c to d in n steps, bounded by x = 0, y = m and limited not to hit the boundaries. The frequency of lattice paths in $B_{c,d}^{(c)}$ is $\left|B_{c,d}^{(c)}(m)\right|$ [1].

Corollary 1

$$Q_{c,d}^{(n)}(m)=\frac{2}{m}p\frac{1-c}{2}q\frac{1+J}{2}\sum_{u=1[\ ]}^{m}\left[2\sqrt{pq}\cos\left(\frac{u\pi}{m}-1\right)\right]^{-1}\sin\left(\frac{cu\pi}{m}\right)\sin\left(\frac{u\pi}{m}\right)\left[2\sqrt{pq}\cos\left(\frac{u\pi}{m}\right)\right]^{n}, \quad (32)$$

$1\leq c\leq m-1$

Proof:

$$P_{c,d}^{(n)}=\frac{2}{m}p\frac{d-c}{2}q\frac{c-d}{2}\sum_{u=1}^{m}\sin\left(\frac{cu\pi}{m}\right)\sin\left(\frac{u\pi}{m}\right)\left[2\sqrt{pq}\cos\left(\frac{u\pi}{m}\right)\right]^{n} \quad (33)$$

If $1\leq c\leq m-1$, then equation (33) holds:

$$\text{Let } \left|B_{c,d}^{(c)}(m)\right|=\frac{2}{m}\sum_{u=1}^{m}\sin\left(\frac{cu\pi}{m}\right)\sin\left(\frac{du\pi}{m}\right)\left[2\cos\left(\frac{u\pi}{m}\right)\right]^{n} \quad (34)$$

$$Q_{c,0}^{(n)}=\sum_{i=1}^{n-1}p_{c,1}^{(i)}.q, \quad (35)$$

where q is the probability of moving from state i to i-1 losing a unit

Setting d =1 as in equation (35),

$$\begin{aligned}
Q_{c,0}^{(n)}&=\frac{2}{m}p\frac{1-c}{2}q\frac{c-1}{2}\sum_{u=1}^{m}\sin\left(\frac{cu\pi}{m}\right)\sin\left(\frac{u\pi}{m}\right)\left[2\sqrt{pq}\cos\left(\frac{u\pi}{m}\right)\right]^{n}.q\\
&=\frac{2}{m}p\frac{1-c}{2}q\frac{c-1}{2}\sum_{i=1}^{n-1}\sum_{u=1}^{m}\sin\left(\frac{cu\pi}{m}\right)\sin\left(\frac{u\pi}{m}\right)\left[2\sqrt{pq}\cos\left(\frac{u\pi}{m}\right)\right]^{i}.q\\
&=\frac{2}{m}p\frac{1-c}{2}q\frac{c+1}{2}\sum_{u=1}^{m}\sin\left(\frac{cu\pi}{m}\right)\sin\left(\frac{u\pi}{m}\right)\sum_{i=1}^{n-1}\left[2\sqrt{pq}\cos\left(\frac{u\pi}{m}\right)\right]^{i}\\
&=\frac{2}{m}p\frac{1-c}{2}q\frac{c+1}{2}\sum_{u=1}^{m}\sin\left(\frac{cu\pi}{m}\right)\sin\left(\frac{u\pi}{mm}\right)\left[1-2\sqrt{pq}\cos\left(\frac{u\pi}{m}\right)\right]^{-1}\left[1-\left\{2\sqrt{pq}\cos\left(\frac{u\pi}{m}\right)\right\}\right]^{n} \quad (36)
\end{aligned}$$

Generating Functions for Game Duration of Gambler's Ruin `

Let m (0 < m < c) be the initial position and $V_{m,n}$ be the probability that the process ends with the nth step at restriction zero. Ultimately, the position after the first step is m+1 or m-1 so that $V_{m,n+1}=pV_{m+1},n+qV_{m-1,n},\quad \forall m,n\geq 0,\ 0<m<c$ (37)

Let the boundary values:

$$V_{o,n} = V_{a,n} = 0; if \ n \geq 1 \tag{38}$$

$$V_{o,o} = 0, \quad V_{m,o} = 0, 0 < m \leq c \tag{39}$$

The generating functions

$$V_m(k) = \sum_{n=0}^{\infty} V_{m,n} k^n, \quad \text{where k is an arbitrary constant.} \tag{40}$$

From equation (37),

$$V_m(k) = pkV_{m+1}(k) + qkV_{m-1}(k), 0 < m < c \tag{41}$$

From equations (38) and (39),

$$V_o(k) = 1, V_c(k) = 0 \tag{42}$$

Substituting the solutions $V_m(k) = \alpha^m(k)$ into equation (41), $\alpha(k)$ satisfies:

$$\alpha(k) = pk\alpha^2(k) + qk \tag{43}$$

Equations (43) has two roots: $\alpha_1(k) = \dfrac{1+\sqrt{1-4pqk^2}}{2pk}$, $\alpha_2(k) = \dfrac{1-\sqrt{1-4pqk^2}}{2pk}$

$$=> Vm(k) = L(k)\alpha_1^m(k) + M(k)\alpha_2^m(k) \tag{44}$$

To validate the boundary conditions in equation (42), $L(k) + M(k) = 1 \ and$ $L(k)\alpha_1^a(k) + M(k)\alpha_2^a(k)$.

Thus, $$V_m(k) = \frac{\alpha_1^a(k)\alpha_2^m(k) - \alpha_1^m(k)\alpha_2^a(k)}{\alpha_1^a(k) - \alpha_1^a} \tag{45}$$

The generating function of the probability of ruin at zero absorption and with nth trial is:

$$V_m(k) = \left(\frac{q}{p}\right)^m \frac{\alpha_1^{a-m}(k) - \alpha_2^a(k)}{\alpha_1^{a-m}(k) - \alpha_2^a(k)} \tag{46}$$

The generating function for the probability of ruin at a absorption is

$$V_m(k) = \frac{\alpha_1^m(k) - \alpha_2^m(k)}{\alpha_1^a(k) - \alpha_2^a(k)} \tag{47}$$

The generating function for game duration is therefore given as

$$V_m(k)=\left(\frac{q}{p}\right)^m\frac{\alpha_1^{a-m}(k)-\alpha_2^a(k)}{\alpha_1^{a-m}(k)-\alpha_2^a(k)}+\frac{\alpha_1^m(k)-\alpha_2^m(k)}{\alpha_1^a(k)-\alpha_2^a(k)} \tag{48}$$

CONCLUSION

It is evident in this work that a classical gambler's ruin problem could be solved using the concept of difference equations, which governs random walks and derivation of limiting differential equations. The generating function for the expected game duration is based on the generating functions of probabilities of ruin at zero and particular unit absorptions. Ultimately, the probability of gambler's ruin depends on payoff distribution and on the initial wealth of the gambler.

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