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COMPUTATIONAL MODELLING OF BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION ON STOCK PRICES

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ABSTRACT: *This paper presents Black-Scholes (BS) partial differential equation (PDE) and Crank-Nicolson (CN) numerical solutions for the valuation of European call option. In particular Crank-Nicolson finite difference scheme was effectively applied to the transformed boundary value problems of European call option with results obtained. It was observed that Black-Scholes and Crank-Nicolson are indistinguishable and errors were most pronounced when sigma were in the range* $0.5 \leq \sigma \leq 1$. *However, this paper presented here has a vital implications to the option traders and in this dynamic area of financial Mathematics.*

KEYWORDS: Crank-Nicolson, Call Option, Stock Prices, Error, Modeling.

INTRODUCTION

This paper is about option pricing and some of its dynamics in financial markets via valuation using Black-Scholes (BS) Partial Differential Equation (PDE) as it explores the changes of option (values) as a function of security and time. Through this PDE the analytical and Crank-Nicolson numerical solutions were found. In the work of Black and Scholes (1973) they obtained a mathematical structure for finding the reasonable price of European options by the use of no-arbitrage principle to describe a PDE which governs the growth of the option price that evolves time to expiration. The details of this options can be found in the following books: Higham (2004), Hull (2003) and Wilmoth et al. (2008) etc. This paper presents a numerical solution to famous Black-Scholes partial differential equation which has a closed form solution of European options; because of its importance researchers has used numerical methods under different conditions to estimate option prices such as Bokes (2011), Yueng (2012) and Sargon (2012) etc. The option under consideration in this study is European call option.

Our interest is to illustrate the use of the model in both analytically and numerically and to identify the sources of relative errors between Black-Scholes and Crank-Nicolson as errors in valuation can cause unreasonable loss for trader Bokes (2011), hence valuation is key properties of trading system. However, the advantage of this proposed method over the previous is determining proper range of sigma when other parameters are constant.

Black-Scholes Model

The seminar paper on option pricing by Black and Scholes (1973) has had great impact on contemporary research in Mathematics of finance. This model is commonly used in financial modeling. The Black-Scholes model is made up of on seven assumptions:

- The asset price has characteristics of a Brownian motion with μ and σ as constants.
- The transaction costs or taxes are not allowed.
- The entire securities are absolutely divisible.
- Dividend is not permitted during the period of the derivatives.
- Unacceptable of riskless arbitrage opportunities.
- The security trading is constant.
- The option is exercised at the time of expiry for both call and put options.

The analytic formula for the prices of European call option is given as follows

$$\left.\begin{aligned} C &= SN(d_1) - Ke^{-rt}N(d_2) \\ d_1 &= \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}\right\} \tag{1}$$

where C is Price of a call option, S is price of underlying asset, K is the strike price, r is the riskless rate, T is time to maturity, σ^2 is variance of underlying asset, σ is standard deviation of the (generally referred to as volatility) underlying asset, and N is the cumulative normal distribution.

Derivation of Black-Scholes (BS) Partial Differential Equation (PDE)

According to Rinalini (2006), the derivation of BS PDE is based on Itô process with an assumption that the stock prices follow a geometric Brownian motion, i.e.

$$dS = \mu S dt + \sigma S dx \tag{2}$$

where S is the stock price, μ is the drift, σ is the volatility of underlying asset and dx is a Wiener process.

Suppose we have an option whose worth $V(S,t)$ depends only on S and t . Assume also that the asset price is varied by a small amount of change dS , then the function V will also be affected. By means of Itô lemma

$$dV = \sigma S \frac{\partial V}{\partial S} dx + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt \quad (3)$$

According to Yueng (2012), the value of one portfolio having one stock can be expressed with the function $V(S,t)$.

$$\pi = V - \Delta S \quad (4)$$

The change in the portfolio at time dt in (4) is given by

$$d\pi = dV - \Delta S \quad (5)$$

Putting (1), (2) into (4), we find that π follows random walk given by.

$$d\pi = \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dx + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt$$

To eliminate the random component in this random walk, let

$$\Delta = \frac{\partial V}{\partial S} \quad (6)$$

Note that Δ is the value of $\frac{\partial V}{\partial S}$ at the start of the time step dt. This results in a portfolio whose increment is wholly deterministic so that

$$d\pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \quad (7)$$

Now that the portfolio is riskless it should earn riskless return. The change in the portfolio at time dt becomes (after substituting (1) and (2) into (4) and dividing through by dt)

$$d\pi = r\pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt ,$$

and

$$r(V - \Delta S) dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

which implies that

$$rV - rS\frac{\partial V}{\partial S} = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

This gives the solution

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{8}$$

This is the Black-Scholes partial differential equation as established.

European Call Option

The BS PDE for European Call Option with value $C(S,t)$ is given in the following equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{9}$$

The following initial and boundary conditions for Call Option:

$$\left.\begin{aligned} C(0,t) &= 0 \\ C(S,t) & \qquad = S \;\; when \;\; S \to \infty \\ C(S,T) &= \max(S-K,0) \end{aligned}\right\} \tag{10}$$

The Crank-Nicolson Finite Difference Method for an Option Pricing Model

The Crank-Nicolson finite difference method is to conquer the stability short-comings by applying the stability and convergence restrictions of the explicit finite difference methods. It is essentially an average of the implicit and explicit methods. However, to implement Crank-Nicolson approximation scheme on Black-Scholes partial differential equation, there must be a price time mesh in order to enhance efficiency as solution exits, the vertical axis in the mesh denotes the stock prices, while the horizontal axis denotes time. Therefore, every grid point in the mesh denotes a horizontal index i and a vertical index j such that every point in the mesh is the option price for a distinct time and a distinct stock price. At every time in the mesh $j\Delta s$ is equivalent to the stock price, and $i\Delta t$ is equivalent to the time. There exist boundary conditions which help in the numerical calculations; by means of the pay-off function. The maturity period, $t = T$ and the option are well computed for all the different initial stock prices using a boundary conditions for uniqueness of solution. To get the prices at $t = 0$, the model solves backwards for every time step from $t = T$, Sargon (2017).

Recall that the Black-Scholes partial differential equation (9).

Let a function $V(S,t)$ in two-dimensional grid points, that is to say i and j stands for the index for stock price, s and time, t respectively. The function $V(S,t)=V_i^j$ can be stated as follows in the subsequent difference scheme Hull (2003).

$$Z_i^{\ j} = \frac{1}{2}\sigma^2 S^2 DSS + rS_i DS - rV_i^{\ j} \tag{11}$$

where DS is the first partial derivative and DSS is the second partial derivative of equation (3.5.6).

$S = i\Delta S$, for $0 \le i \le m$, $t = j\Delta t$ for $0 \le j \le i$

and

$$DSS = \frac{V_{i+1}^j - 2V_i^{\ j} + V_{i-1}^j}{\Delta^2} \tag{12}$$

$$DS = \frac{V_{i+j}^j - V_{i-1}^j}{2\Delta S} \tag{13}$$

Taking forward difference and backward difference approximations respectively yields implicit and explicit schemes given below.

If we use a forward difference approximation to the time partial derivative, we obtain explicit scheme

$$\frac{V_i^{\ j+1} - V_i^{\ j}}{\Delta t} + Z_i^j = 0 \tag{14}$$

and similarly, we obtain the implicit scheme

$$\frac{V_i^{\ j+1} - V_i^{\ j}}{\Delta t} + Z_i^{j+1} = 0 \tag{15}$$

The averages of equations (14) and (15) yields Crank-Nicolson method of approximation

$$\frac{V_i^{\ j+1} - V_i^{\ j}}{\Delta t} + \frac{1}{2}\left(Z_i^j + Z_i^{j+1}\right) = 0 \tag{16}$$

From equation (16)

$$V_i^{\ j} - \frac{\Delta t}{2} Z_i^j = V_i^{\ j+1} + \frac{\Delta t}{2} Z_i^{j+1} \tag{17}$$

where

$$Z_i^j = \frac{\sigma^2 S_i^2}{2(\Delta S)^2}\left[V_{i+1}^j - 2V_i^j + V_{i-1}^j\right] + \frac{rS_i}{2\Delta S}\left[V_{i+1}^j - V_{i-1}^j\right] - rV_i^j$$

and

$$Z_i^{j+1} = \frac{\sigma^2 S_i^2}{2(\Delta S)^2}\left[V_{i+1}^{j+1} - 2V_i^{j+1} + V_{i-1}^{j+1}\right] + \frac{rS_i}{2\Delta S}\left[V_{i+1}^{j+1} - V_{i-1}^{j+1}\right] - rV_i^{j+1}$$

Collecting like terms of V_{i-1}, V_i *and* V_{i+1} and simplifying gives

$$Z_i^j = V_{i-1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta tS_i}{2\Delta S}\right] + V_i^j\left[\frac{2\Delta t}{\Delta t} + \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right] - V_{i+1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta tS_i}{2\Delta S}\right] \tag{18}$$

and

$$Z_i^{j+1} = V_{i-1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta tS_i}{2\Delta S}\right] + V_i^{j+1}\left[\frac{2\Delta t}{\Delta t} - \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t}{2}\right] + V_{i+1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta tS_i}{2\Delta S}\right] \tag{19}$$

Using (18) and (19) solving simultaneously and taking the average of these two equations we obtain

$$\left.\begin{aligned} &V_{i-1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} - \frac{r\Delta tS_i}{4\Delta S}\right] + V_i^j\left[1 + \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right] - V_{i+1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} + \frac{r\Delta tS_i}{4\Delta S}\right] \\ &= V_{i-1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} - \frac{r\Delta tS_i}{4\Delta S}\right] + V_i^{j+1}\left[1 - \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t}{2}\right] + V_{i+1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} + \frac{r\Delta tS_i}{4\Delta S}\right] \end{aligned}\right\} \tag{20}$$

The expressions inside the square brackets of (20) are replaced with the a_i, b_i, c_i . We obtain the following equation

$$a_i V_{i-1}^j + (1 + b_i)V_i^j - c_i V_{i+1}^j = a_i V_{i-1}^{j+1} + (1 - b_i)V_i^{j+1} + c_i V_{i+1}^{j+1} \tag{21}$$

where

$a_i = \frac{\Delta t}{4}\left(\sigma^2 S_i^2 - rS_i\right), b_i = -\frac{\Delta t}{2}\left(\sigma^2 S_i^2 - r\right)$ and $c_i = \frac{\Delta t}{4}\left(\sigma^2 S_i^2 + rS_i\right)$.

a_i, b_i, c_i are random variables; $i = 0,1,\ldots,M$.

The CN discretization out come in tridiagonal matrix which is solvable at each time step.

RESULTS AND DISCUSSION

Table 1: Comparing the performance of the Black-Scholes exact values and Crank-Nicolson finite difference method for European Call Option when the initial stock prices are 40 and 50 with K = 25, r = 0.2 and T = 1

Sigma	$S_o = 40$,	$K = 25$,	r=0.2	$S_o = 50$,	$K = 25$,	r=0.2,
	BS Exact values	CN	Relative Error	BS Exact Values	CN	Relative Error
0.25	19.5398	19.5398	1.0236E-04	29.5321	29.4815	1.7134E-03
0.3	19.5695	19.5695	6.6941E-04	29.5357	29.3841	5.1328E-03
0.35	19.6371	19.6371	2.266E-04	29.5506	29.2378	0.0106
0.4	19.7508	19.7508	5.2656E-03	29.5877	29.0650	0.0177
0.45	19.9117	19.9117	9.7028E-03	29.6565	28.8896	0.0259
0.5	20.1167	20.1167	0.0154	29.7625	28.7302	0.0347
0.55	20.3607	20.3607	0.0220	29.9075	28.5990	0.0438
0.6	20.6383	20.6383	0.0293	30.0906	28.5022	0.0528
0.65	20.9441	20.9441	0.0369	30.3094	28.4420	0.0616
0.7	21.2733	21.2733	0.0447	30.5604	28.4178	0.0701
0.75	21.6219	21.6219	0.0525	30.8401	28.4269	0.0782
0.8	21.9861	21.9861	0.06013	31.1446	28.4655	0.0860
0.85	22.3630	22.3630	0.06763	31.4707	28.5293	0.0935
0.9	22.7497	22.7497	0.07495	31.8150	28.6141	0.1006
0.95	23.1442	23.1442	0.08212	32.1746	28.7154	0.1075
1.0	23.5443	23.5443	0.08913	32.5470	28.8296	0.1142

Values in Table 1 were generated by fixing r=0.2, k=25 while allowing S_0 and σ vary in (1) and (20) for call options such that $s_0 = 40 \text{ and } 50$, $\sigma = (0.25, 0.30, ..., 1.00)$ presented in column 1 are the difference values of σ ,columns 2 and 3 are the exact values of BS and CN respectively. The 4th column gives the relative error (RE) of the difference between the estimated prices using BS and CN pricing schemes given by $|BS - CN| / BS$. RE is the ratio of absolute difference between BS and CN to BS such that when this ratio is very small, the performances of both BS and CN are equivalent, otherwise they are noticeable difference as can be observed in Table 1.The difference in performance increases from near zero with increasing value of σ. Visual inspection show that BS and CN are indistinguishable when $0 < \sigma < 0.5$ and the differences starts when σ is in the range $0.5 \le \sigma \le 1$.

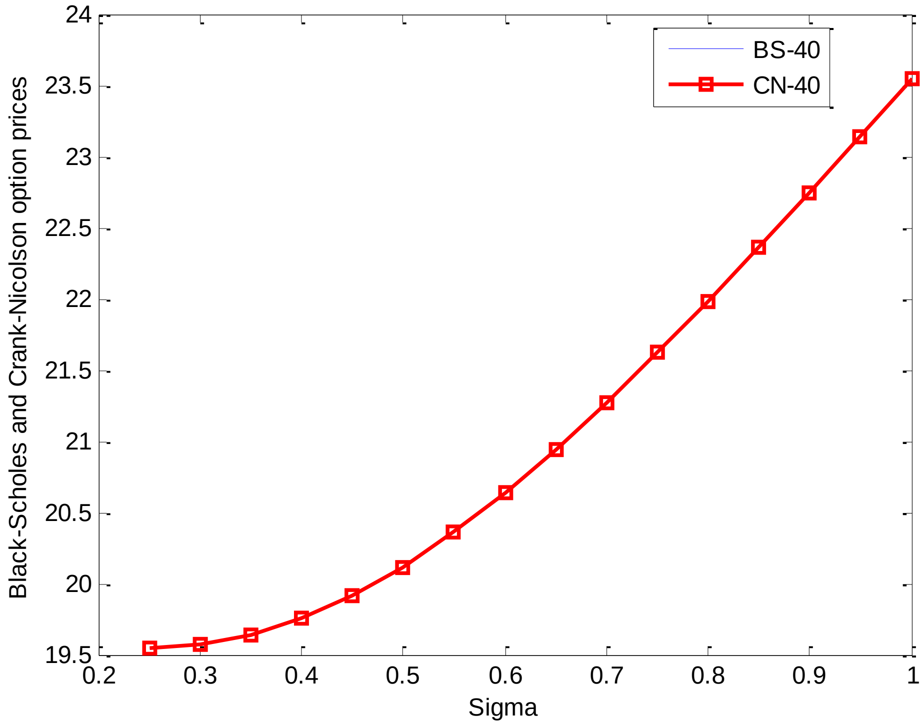

Figure 1: Comparing Black-Scholes with Crank-Nicolson Numerical Solution under Initial Stock Price of 40 for Call Option.

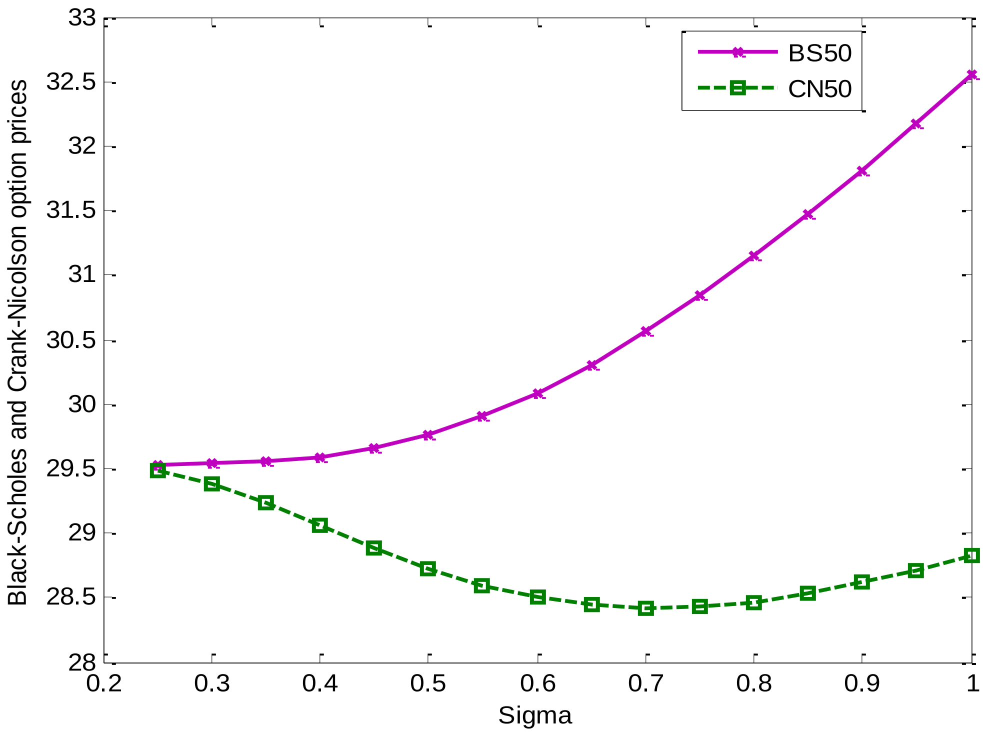

Figure 2: Comparing the Performance of the Black-Scholes Exact Values and Crank-Nicolson Finite Difference Method for European Call Option when Initial Stock Price is 50

In Figure 1,BS and CN are indistinguishable even with variations of sigma which also agree with Table 1. Whereas in Figure 2, there were noticeable differences when the initial stock prices increased to 50 and various variations of sigma within the stipulated trading days.

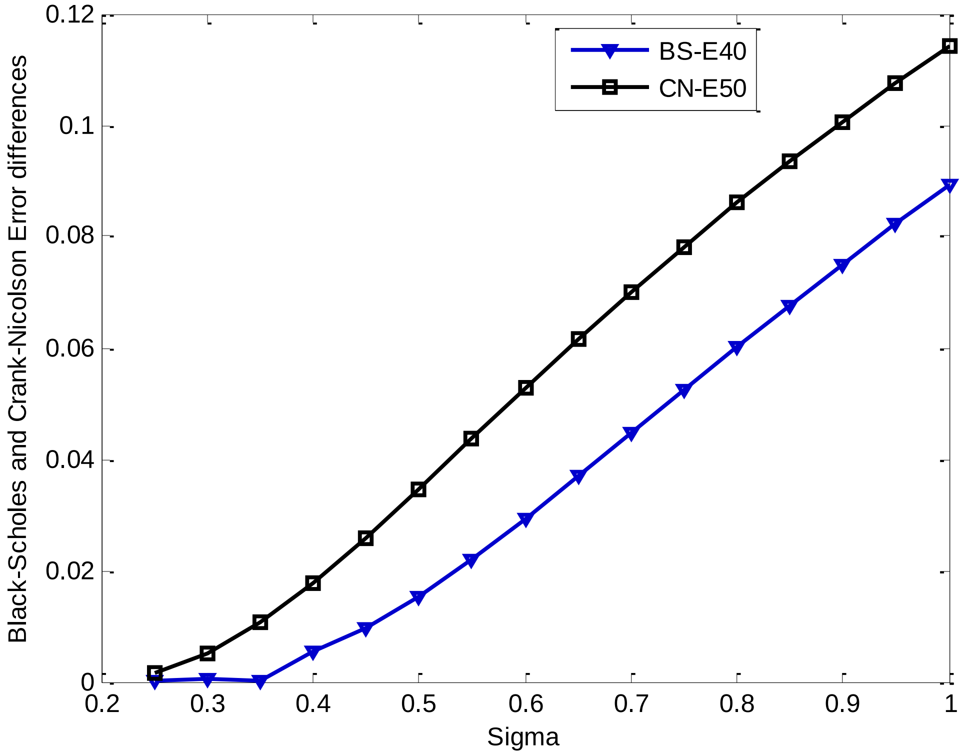

Figure 3: Levels of Relative Errors of Two Initial Stock Prices for Call Option.

Figure 3 this shows the error differences with different initial stock prices as against different variations of sigma. These also agree with our explanations in Table1

CONCLUSION

This paper considered Black-Scholes partial differential equation, its variants and interpretation of results following the methods in Section 3. From the results of BS and CN numerical solutions, it was noticed that BS and CN are indistinguishable and errors were most pronounced when sigma (σ) is in the range of $0.5 \leq \sigma \leq 1$. Obviously, it is not advisable to go beyond the stipulated range of sigma (σ); in other to avoid much difference.

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