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ESTIMATION OF STOCK PRICES USING BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION FOR PUT OPTION

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ABSTRACT: *This paper is centred on the implementation of Black-Scholes (BS) partial differential equation (PDE) via Crank-Nicolson finite difference method for the valuation of European put option. First, we explained how Black-Scholes (BS) partial differential equation and Crank-Nicolson (CN) numerical solutions can be used to estimate option prices and then we derived the method and results obtained through our simulation studies. From the results, it was discovered that Black-Scholes analytical and CN numerical values were relatively close but increase in the variation of sigma introduced much errors in the pricing analysis; with this increase in sigma profit making within the trading days is not possible.*

KEYWORDS: Black-Scholes PDE, Numerical Solutions, Put Option, Simulation.

INTRODUCTION

An option is a tool whose worth is derived from the principal asset which is otherwise known as financial derivative. This type of derivative does have anything in common with mathematical meaning of derivative. In other words, an option on underlying asset is a business between parties who come together to agree on either buying or selling an underlying asset at a determined strike price in the future for a fixed price. The cost of the option lies on the underlying asset, which is usually a stock. The holder has the right but cannot be compelled to buy, for call option where European put option involves the ability to sell an asset for a certain charge at a prescribed date in the future.

However, the use of Black-Scholes equation in valuation of options has attracted many research workers such as Black and Scholes(1973), Hull (2003), Macbeth and Merville (1979), Nwobi et al. (2019), Razali (2006), Rinalini (2006), Wilmoth et al.(2008) and Yueng (2017) etc.

In this paper, we shall be interested in Crank-Nicolson (CN) finite difference method for valuation of European put option which have gained the interest of researchers for finding approximate solutions to PDEs and this interest is driven by demand of applications of societal problems. Secondly, the influences of various variations of sigma as other parameters in the model remain fixed.

Modeling Underlying Asset Price

In financial markets, investors and financial analysts are generally too interested on how to maximize profit over particular trading days, that is, the changes in the price of goods and

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services. Therefore, modeling a behavior of a stock exchange market can be made through its relative change of the unstable market variables in time so as to predict stock price fluctuation, advice investors and corporative owners who are working out for convenient ways to do business by issuing of stocks in their corporations. Now, the market price behavior shows the characteristics as a stochastic process called “Brownian motion” or Wiener process with drift. It is an important example of stochastic processes satisfying a Stochastic Differential Equation (SDE) displayed in Figure 1.

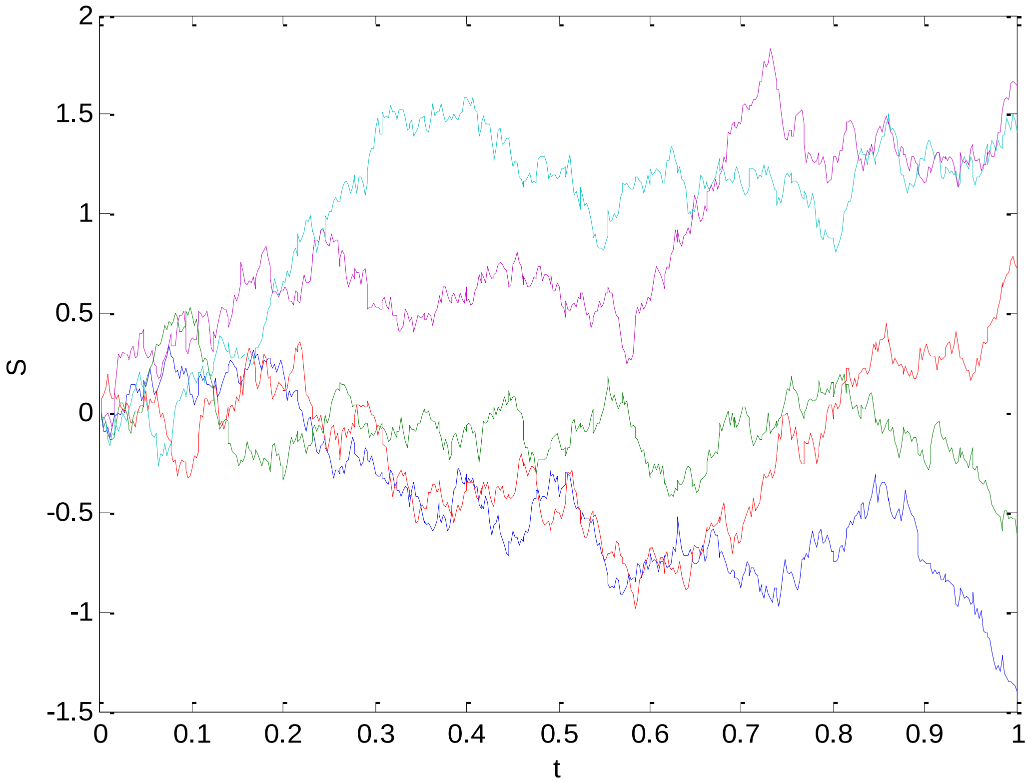

Figure 1: Sample trajectories of the stock price process following Black-Scholes Model

Black-Scholes Model

In the work of Black and Scholes (1973) the following seven assumptions are made:

- The asset price has properties of a Brownian motion with μ and σ as constants.
- There are no transaction costs or taxes.
- All securities are perfectly divisible.
- There is no dividend during the life of the derivatives.
- There are no riskless arbitrage opportunities.
- The security trading is continuous.
- The option is exercised at the time of maturity for both call and put options.

The analytic formula for the prices of European put option is given as follows

$$\left.\begin{aligned} P &= SN\left(d_1\right) - Ke^{-rt}N\left(d_2\right) \\ & \frac{In\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ & d_2 = d_1 - \sigma\sqrt{T} \end{aligned}\right\} \tag{1}$$

where p is Price of a call option, S is price of underlying asset, K is the strike price, r is the riskless rate, T is time to maturity, σ^2 is variance of underlying asset, σ is standard deviation of the (generally referred to as volatility) underlying asset, and N is the cumulative normal distribution.

Derivation of Black-Scholes (BS) Partial Differential Equation (PDE)

According to Rinalini (2006), the derivation of BS PDE is based on Itô process with an assumption that the stock prices follow a geometric Brownian motion, i.e.

$$dS = \mu S dt + \sigma S dx \tag{2}$$

where S is the stock price, μ is the drift, σ is the volatility of underlying asset and dx is a Wiener process.

Suppose we have an option whose worth $V(S,t)$ depends only on S and t . Assume also that the asset price is varied by a small amount of change dS , then the function V will also be affected. By means of Itô lemma

$$dV = \sigma S\frac{\partial V}{\partial S}dx + \left(\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt \tag{3}$$

According to Yueng(2012), the value of one portfolio having one stock can be expressed with the function $V\left(S,t\right)$.

$$\pi = V - \Delta S \tag{4}$$

The change in the portfolio at time dt in (4) is given by

$$d\pi = dV - \Delta S \tag{5}$$

Putting (1), (2) into (4),we find that π follows random walk given by.

$$d\pi = \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right)dx + \left(\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\Delta S\right)dt$$

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To eliminate the random component in this random walk, let

$$\Delta = \frac{\partial V}{\partial S} \tag{6}$$

Note that Δ is the value of $\frac{\partial V}{\partial S}$ at the start of the time step dt. This results in a portfolio whose increment is wholly deterministic so that

$$d\pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \tag{7}$$

Now that the portfolio is riskless it should earn riskless return. The change in the portfolio at time dt becomes (after substituting (1) and (2) into (4) and dividing through by dt)

$$d\pi = r\pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt,$$

and

$$r(V - \Delta S) dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

which implies that

$$rV - rS \frac{\partial V}{\partial S} = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

This gives the solution

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \tag{8}$$

This is the Black-Scholes partial differential equation as established.

European Put Option

The BS PDE for European put option with value $P(S,t)$ is defined as in (8)

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0 \tag{9}$$

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With the following initial and boundary conditions:

$$\left.\begin{array}{c} P(0,t) = Ke^{-rt} \\ P(S,t) = 0 \ when \ S \to \infty \\ P(S,T) = \max(K-S,0) \end{array}\right\} \tag{10}$$

The Crank-Nicolson Finite Difference Method for an Option Pricing Model

The Crank-Nicolson finite difference method is to surmount the stability short-comings by applying the stability and convergence restrictions of the explicit finite difference methods. It is essentially an average of the implicit and explicit methods. However, to implement Crank-Nicolson approximation scheme on Black-Scholes partial differential equation, there must be a price time mesh in order to enhance efficiency as solution exits, the vertical axis in the mesh denotes the stock prices, while the horizontal axis denotes time. Therefore, every grid point in the mesh denotes a horizontal index i and a vertical index j such that every point in the mesh is the option price for a distinct time and a distinct stock price. At every time in the mesh $j\Delta s$ is equivalent to the stock price, and $i\Delta t$ is equivalent to the time. There exist boundary conditions which help in the numerical calculations; by means of the pay-off function. The maturity period, $t = T$ and the option are well computed for all the different initial stock prices using a boundary conditions for uniqueness of solution. To get the prices at $t = 0$, the model solves backwards for every time step from $t = T$, Sargon (2017).

Recall that the Black-Scholes partial differential equation (8).

Let a function $V(\mathrm{S},t)$ in two-dimensional grid points, that is to say i and j stands for the index for stock price, s and time, t respectively. The function $V(S,t) = V_i^j$ can be stated as follows in the subsequent difference scheme [2].

$$Z_i^j = \frac{1}{2}\sigma^2 S^2 DSS + rS_i DS - rV_i^j \tag{11}$$

where DS is the first partial derivative and DSS is the second partial derivative of equation(11).

$S = i\Delta S$, for $0 \le i \le m$, $t = j\Delta t$ for $0 \le j \le i$

and

$$DSS = \frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{\Delta^2} \tag{12}$$

$$DS = \frac{V_{i+j}^j - V_{i-1}^j}{2\Delta S} \tag{13}$$

Taking forward difference and backward difference approximations respectively yields implicit and explicit schemes given below.

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If we use a forward difference approximation to the time partial derivative, we obtain explicit scheme

$$\frac{V_i^{j+1}-V_i^j}{\Delta t}+Z_i^j=0 \tag{14}$$

and similarly, we obtain the implicit scheme

$$\frac{V_i^{j+1}-V_i^j}{\Delta t}+Z_i^{j+1}=0 \tag{15}$$

The averages of equations (14) and (15) yields Crank-Nicolson method of approximation

$$\frac{V_i^{j+1}-V_i^j}{\Delta t}+\frac{1}{2}\left(Z_i^j+Z_i^{j+1}\right)=0 \tag{16}$$

From equation (16)

$$V_i^j-\frac{\Delta t}{2}Z_i^j=V_i^{j+1}+\frac{\Delta t}{2}Z_i^{j+1} \tag{17}$$

where

$$Z_i^j=\frac{\sigma^2 S_i^2}{2\left(\Delta S\right)^2}\left[V_{i+1}^j-2V_i^j+V_{i-1}^j\right]+\frac{rS_i}{2\Delta S}\left[V_{i+1}^j-V_{i-1}^j\right]-rV_i^j$$

and

$$Z_i^{j+1}=\frac{\sigma^2 S_i^2}{2\left(\Delta S\right)^2}\left[V_{i+1}^{j+1}-2V_i^{j+1}+V_{i-1}^{j+1}\right]+\frac{rS_i}{2\Delta S}\left[V_{i+1}^{j+1}-V_{i-1}^{j+1}\right]-rV_i^{j+1}$$

Collecting like terms of V_{i-1}, V_i *and* V_{i+1} and simplifying gives

$$Z_i^j=V_{i-1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{2\left(\Delta S\right)^2}-\frac{r\Delta tS_i}{2\Delta S}\right]+V_i^j\left[\frac{2\Delta t}{\Delta t}+\frac{\Delta t\sigma^2 S_i^2}{2\left(\Delta S\right)^2}+\frac{r\Delta t}{2}\right]-V_{i+1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{2\left(\Delta S\right)^2}+\frac{r\Delta tS_i}{2\Delta S}\right] \tag{18}$$

and

$$Z_i^{j+1}=V_{i-1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{2\left(\Delta S\right)^2}-\frac{r\Delta tS_i}{2\Delta S}\right]+V_i^{j+1}\left[\frac{2\Delta t}{\Delta t}-\frac{\Delta t\sigma^2 S_i^2}{2\left(\Delta S\right)^2}-\frac{r\Delta t}{2}\right]+V_{i+1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{2\left(\Delta S\right)^2}+\frac{r\Delta tS_i}{2\Delta S}\right] \tag{19}$$

Using (18) and (19) solving simultaneously and taking the average of these two equations we obtain

$$\left.\begin{aligned} &V_{i-1}^{j}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2}-\frac{r\Delta tS_i}{4\Delta S}\right]+V_i^{j}\left[1+\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2}+\frac{r\Delta t}{2}\right]-V_{i+1}^{j}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2}+\frac{r\Delta tS_i}{4\Delta S}\right] \\ &=V_{i-1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2}-\frac{r\Delta tS_i}{4\Delta S}\right]+V_i^{j+1}\left[1-\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2}-\frac{r\Delta t}{2}\right]+V_{i+1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2}+\frac{r\Delta tS_i}{4\Delta S}\right]\end{aligned}\right\} \tag{20}$$

The expressions inside the square brackets of (20) are replaced with the a_i, b_i, c_i. We obtain the following equation

$$a_iV_{i-1}^{j}+(1+b_i)V_i^{j}-c_iV_{i+1}^{j}=a_iV_{i-1}^{j+1}+(1-b_i)V_i^{j+1}+c_iV_{i+1}^{j+1} \tag{21}$$

where

$a_i=\frac{\Delta t}{4}\left(\sigma^2S_i^2-rS_i\right), b_i=-\frac{\Delta t}{2}\left(\sigma^2S_i^2-r\right)$ and $c_i=\frac{\Delta t}{4}\left(\sigma^2S_i^2+rS_i\right)$.

a_i, b_i, c_i are random variables; $i=0,1,\ldots,M$.

The CN discretization outcome in tridiagonal matrix which is solvable at each time step.

RESULTS AND DISCUSSIONS

Table 1. Comparing the performance of the Black-Scholes exact values and Crank-Nicolson finite difference method for European Put Option when initial stock prices are 40 and 50 with K = 100, r = 0.2 and T = 1

Sigma Sigma	S_o = 40, K = 100			S_o = 50, K = 100,		
	BS Exact values	CN	Relative Error	BS Exact Values	CN	Relative Error
0.25	41.8817	41.8778	9.3119E-05	32.0183	31.9605	1.8052E-03
0.3	41.9211	41.8977	5.5819E-04	32.2717	32.1011	5.2864E-03
0.35	42.0213	41.9458	1.7967E-03	32.6694	32.3076	0.01107
0.4	42.2025	42.0283	4.1277E-03	33.1966	32.5633	0.01908
0.45	42.4721	42.1446	7.7109E-03	33.8322	32.8510	0.029002
0.5	42.8277	42.2898	0.01256	34.5553	33.1562	0.04049
0.55	43.2622	42.4578	0.01859	35.3479	33.4685	0.05317
0.6	43.7659	42.6422	0.02568	36.1951	33.7805	0.06671
0.65	44.3292	42.8374	0.03365	37.0850	34.0870	0.08084
0.7	44.9429	43.0389	0.04236	38.0079	34.3849	0.09532
0.75	45.5988	43.2432	0.05166	38.9559	34.6722	0.10996
0.8	46.2893	43.4474	0.06139	39.9226	34.9476	0.1246
0.85	47.0083	43.6494	0.07145	40.9028	35.2108	0.1392
0.9	47.7501	43.8478	0.08172	41.8921	35.4616	0.1535
0.95	48.5099	44.0415	0.09211	42.8869	35.7001	0.1676
1.0	49.2837	44.2297	0.10255	43.8839	35.9265	0.1813

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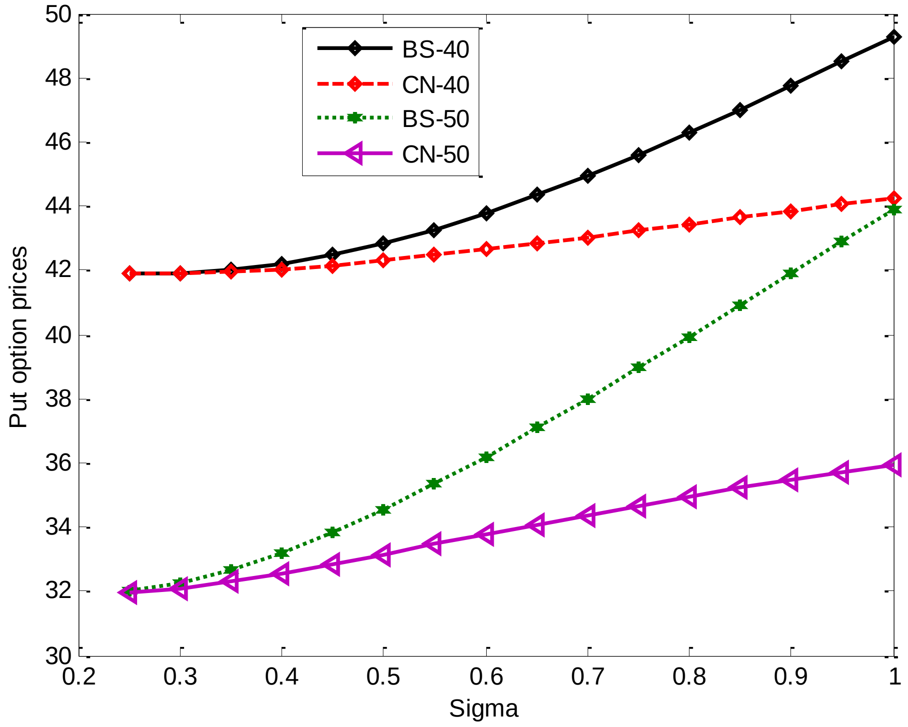

Figure 1: Comparing Black-Scholes with Crank-Nicolson Numerical Solution under Initial Stock Prices of 40 and 50 for Put Option.

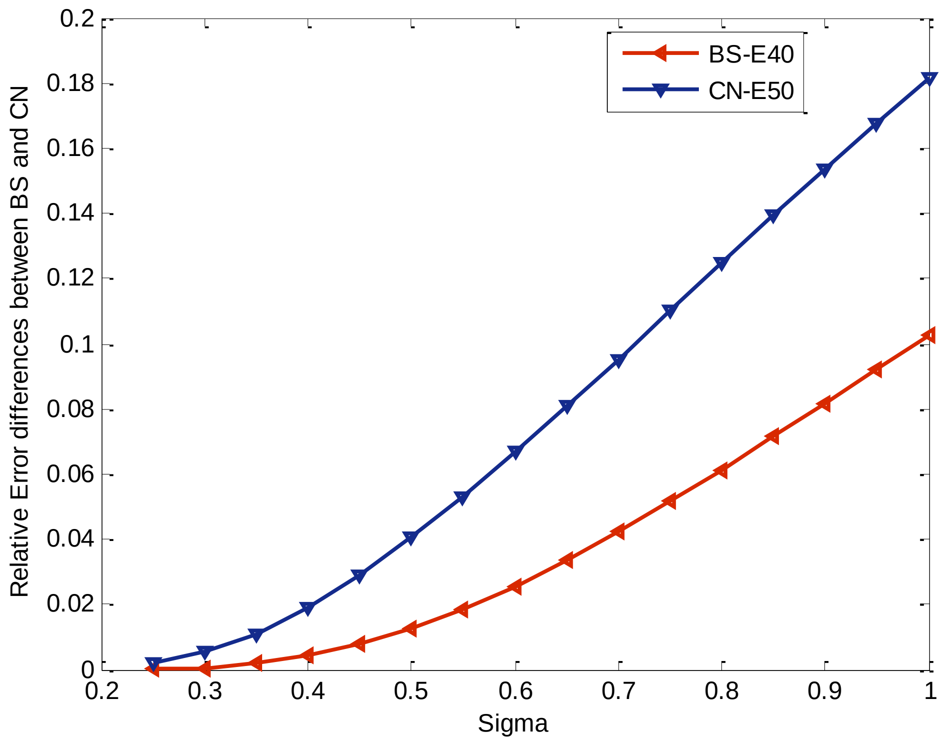

Figure 2: Levels of Relative Errors of Different Initial Stock Prices for Put Option.

Figure 2 this displays the error differences with different initial stock prices as against different variations of sigma. These also agree with our interpretations in Table 1.

The Values in Table 1 were obtained by setting up r=0.2, k=100 while allowing S_0 and σ conform in (1) and (20) for put options such that $s_0 = 40 \, and \, 50 \quad \sigma = (0.25, 0.30, ..., 1.00)$ as shown in column 1 are the difference values of σ ,columns 2 and 3 are the analytical values of BS and CN numerical solutions respectively. The 4th column gives the relative error (RE) of the difference connecting the estimated prices using BS and CN pricing methods given by $|BS - CN| / BS$. RE is the proportion of absolute difference between BS and CN to BS such that when this proportion is very little, the performances of both BS and CN are comparable, or else they are obvious difference as can be observed in Table 1.The difference in performance increases from close to zero with increasing value of σ. Visual assessment show that BS and CN are identical when $0 < \sigma < 0.5$ and the difference starts when σ is in the range $0.5 \le \sigma \le 1$ for European put option. This can also be seen in Figure 1.

CONCLUSION

This paper has two linked topics namely: Black-Scholes Partial differential equation and the method of finite difference method respectively. It has established that the finite difference method is helpful and relatively easy way to approximate solutions of Black-Scholes partial differential equation. A brief analysis showed that errors were most pronounced, when sigma (σ) is in the range $0.5 \le \sigma \le 1$.

Therefore, our novel contribution is unique, efficient and reliable in pricing European put option.

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