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DETERMINATION OF THE DEGREE OF HOMOGENEITY OF PRIMITIVE PERMUTATION GROUPS VIA THE SOCLE OF THE GROUPS

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ABSTRACT: *The homogeneity of finite primitive groups was based on the O'Nan-Sott theorem and classification of finite primitive groups. The determination of the degree of homogeneity using the concept of the socle of a group is as a result of the complexity in the geometric approach, where dominantly the groups were subjection through portioning. In an effort to use the analytical approach emphasis was given to the invariant properties of the minimal normal subgroups, as it is unique or at most two. Further investigation showed that the socle of homogeneous groups were homogeneous except for which* $soc(G) = 1$.

KEYWORDS: Socle, homogeneous, Proups, Finite, Primitive, Permutation.

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INTRODUCTION

The term socle is coined to suit the idea of a subgroup generated by minimal normal subgroups. The early research carried out on homogeneity was based on the work of Livingstone and Wagner (1965) and was further carried out by Kantor (1972) in which it was proved that a group is k-homogeneous if it is k-transitive in k-subsets of a permutated points. The idea was further captured in Dixon and Mortimer (1996). The work of Kantor (1972) used affine and projective linear groups to ascertain his results. Another notable contribution on the study of k-homogeneity was due the work of Martin and Segan (2000) and Cameron (2000), in which they used partition and the t-orbit respectively to view the concept and further proved some results. The socle of a group comes as result of the classification scheme for finite primitive groups due to O'Nan-Scott (1979), where the classification was carried out on the group G which is a finite primitive group and that the subgroup H is the socle of the group. Thus the socle of a group is a very vital tool in the study of homogeneity of finite primitive permutation groups. The concept of the socle will further pave way for the analytical approach in the determination of the degree of homogeneity of finite primitive permutation groups.

Preliminary results.

In this section we give basic results which will be useful later; we start with the definition of a socle of a group, followed by the existence of minimal normal subgroups which is unique or at most two. Also series of results were provided to furnish us in the determination of the degree of homogeneity via the socle of the groups.

Definition 2.1; The socle of G of a finite group is the subgroup generated by the minimal normal subgroups of G denoted by $soc(G)$.

It follows that that the minimal normal subgroups are at most two or unique thus:

Theorem 2.2: The minimal normal subgroups of a group is unique or at most two.

Proof: the result is true if G have $soc(G) = 1$, hence suppose that G has more two, say G has c_1, c_2 and c_3. Then each pair of the c_i intersect trivially and thus the two of them are contained in the centralizer of the third therefore $c_2, c_3 \leq C_G(c_1)$ and it follows that c_2, c_3 and $C_G(c_1)$ are regular and hence equal. Hence by orbit-stabiliser theorem they have at most two distinct socle. Suppose $c_1, and\ c_2$ are distinct socle of G then $c_1 = C_G(c_2)$ and $c_2 = C_G(c_1)$ and they are both regular on Ω. Thus $N_{sym(\Omega)}(C)$ on Ω is permutatively isomorphic to $Hol(C)$, therefore from the permutation isomorphism and the involution in $sym(C)$ we have $C_L \times C_R \leq G \leq Hol(C)$. Which we have Inn(C)$\leq G \leq Aut(C)$.

Remark 2.3

(i) the socle of a group are pair wise disjoint and they commute in pairs
(ii) the socle of a finite primitive group is more restricted and any minimal normal subgroup of itself is a direct product of isomorphic simple groups.
With the conditions given above, we define the following key terms

Definition 2.4: Let $G \leq sym(\Omega)$ the block Δ is called the minimal block for G if Δ contains at least two points and no other block with at least two points is properly contained in Δ.

Remark 2.5: Every transitive group of degree at least two possess a minimal blocks.

Definition 2.6: A group G is primitive if the only blocks are the trivial ones, or rather a group is primitive if and only if it is of prime degree.

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Definition 2.7: Let $\alpha \in G$, then the stabiliser of α is given by $G_\alpha = \{g \in G | \alpha^g = \alpha..$

Definition 2.8: Let $\alpha \in G$, orbit of G containing α is given by $\alpha^G = \{\alpha^g | g \in G.$ $= \{g^{-1}\alpha g | g \in G\}$

Theorem 2.9: Let G be a finite group, then $soc(G) \leq N_G(H)$ for each normal subgroup H of G.

This above theorem is line with the work of Wielandt (1969) on permutation groups through invariant relations and functions. Therefore, socle groups are subnormal and invariant.

Remark 2.10

We can say therefore, if G has a minimal normal subgroup, K say, then
(i) for some prime p and some integer d, then G is a regular elementary abelian group of order p^d and $soc(G) = \text{K} = C_G(K)$.
(ii) K is non abelian, where $C_G(K) = 1$ and $soc(G) = K$.

The next theorem is due Jordan-Witt, the theorem will provide us with the condition for the existence of invariant subgroups.

Lemma 2.11**:** Let $G \leq sym(\Omega)$ be k −transitive or some $k \geq 1$, let Δ be a subset of Ω with $|\Delta| = k$. put $H = G_{(\Delta)}$ and suppose that $K \leq H$ has the property .

For some $x \epsilon G$ such that $x^{-1}Kx \leq H$, there exist $y \epsilon H$ such that $x^{-1}Kx = y^{-1}Ky$. Let $\Gamma = fix(K) \geq \Delta$, and put $N = N_G(K)$. then Γ is N- invariant and N acts transitively on Ω.

Proof: for each $\delta \epsilon \Gamma$ and $x \epsilon N$, $(\delta^x = \delta^{kx} = \delta^k$, so $\delta^x \epsilon \Gamma$ thus $\Gamma\ is\ N -$ invariant.

Now since G is k-transitive, in order to prove that N acts transitively on Γ it is enough to show that for each $x \epsilon G$ such that $\Delta^x \subset$ there exist z϶N such that xz^{-1} acts trivially on Δ. However, $\Delta^x \subset \Gamma$ implies that xKx^{-1} acts trivially on Δ, and so $xKx^{-11} \leq H$. tus by the statement of the lemma there exist $y \epsilon H$ such that $xKx^{-1} = y^{-1}Ky$ and so $z = yx$.

The next theorem is as a result for the existence of $k -$ transitive subgroup.

Theorem 2.12: Let $G \leq sym(\Omega)$ be a k-primitive group for some integer k≥2 and $|\Omega| \geq 2$ and let H be a nontrivial normal subgroup of G. then either

(i) H is k-transitive , or
(ii) $K = 2$ and H is regular elementary abelain 2-group.

Proof: since G is primitive, certainly transitive. fixα϶Ω. Then $K = G_\alpha$ is (k-1) primitive on $\Omega' = \Omega/\{\alpha\}$, and $M = H_\alpha$ normal in K.

If k=2, then K is primitive on Ω^1, and so either M is trivial or M is transitive. also H is regular and hence it is elementary abelian 2-transitive group.

Now suppose $K \geq 3$ and we proceed with the induction process, if H is not regular then $M \neq 1$. Thus since K is $(k-1) -$ transitive on Ω′, then H is k-transitive.to show that M is regular by definition H must be sharply 2-transitive. choose β϶Ω′ and put $L = G_{\alpha\beta}$, L and $\Omega'' = \Omega'/(\alpha, \beta)$ then L is k-2 primitive on Ω″ thus there exist $z \epsilon H$ such that $(\alpha,\beta)^2 = (\beta,\alpha)$ so z^2 fixes(α,β), therefore $z^2 = 1$. With the commutator which is in H that fixes (α, β) thus $z = 1$, hence z centralizes L and act on Ω″. However the orbit of $\{z\}$ has length 1 or 2 which is of length 1. Since $|\Omega| > 4$, this contradicts the primitivity of L on Ω''. Hence M is not regular.

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It can be seen from the theorem that follows, the socle of a group can best be describe base on the classification of finite simple group.

The next theorem is due to [13] in which the concept of socle purely relied upon.

Theorem 2.13: Let G be a group which acts primitively and faithfully on Ω with $|\Omega| = n$. Let $H = soc(G)$ and $\alpha \in \Omega$. Then H is homogeneous of type T and exactly one of the cases holds.

(i) Affine, T is abelian of order p and $n = p^m$ and G_α is a complement to which acts on H and is simple.
(ii) Almost simple m = 1 and $H \lhd G \leq Aut(k)$
(iii) Diagonal type $m \geq 2$ and $n = |T|^{m-1}$. Also, G is a subgroup of $K = (TW_rS_m) \cdot Out(T) \leq Aut(T)W_rS_m$
(iv) product type $m = rs$
(v) Twisted wreath type, H acts regularly and $n = |T|^m$.

Definition 2.14: A group is said to be almost simple if it has a non abelian simple group and a unique minimal normal subgroups.

MAIN RESULTS

Here we establish results which are useful in the determination of the degree of homogeneity of primitive permutation groups via the socle of the groups.

Theorem 3.1

Let G be a finite homogeneous group then $soc(G)$ is a subgroup of the normalizer of the subgroup H.

Proof
Let G be a k-homogeneous group for $k \geq 1$, then the stabilizer G_x is $(k-1)$ transitive on Ω/x, let the $soc(G) \neq 1$, clearly $1 \in soc(G)$ therefore for every $x \in G$ and $y \in H$ implies $xy \in soc(G) \lhd N_G H$ therefore $xy^{-1} \in N_G(H)$

Theorem 3.2

Let $G \leq Sym(\Omega)$ and let $soc(G) = 1$, then G_α is the only socle of the group G.

Proof
If G is a finite primitive group, then G is simple and the chain of subgroup of G is $1 \lhd G_\alpha \lhd G$. Hence G_α is the only maximal subgroup of G and G_α is either 1 or G, so $soc(G) = G_\alpha$

Theorem 3.3

Let G be a finite group then $soc(G)$ is a normal subgroup of H. Also if H is k-transitive, then G is k-homogeneous.

Proof

It follows from Theorem 3.1 that, the socle of a group G is a subgroup of the normalizer of H. It implies that, for any $K \leq H$ is normal in G and so K is minimal. Therefore it is contained in the normalizer of H in G. Next we prove that it is transitive. This follows from Theorem 3.13 and also in line with Theorem 3.1. Hence H is k-transitive and so in consequence of Theorem 3.13, it imply G is k-homogenous.

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Theorem 3.4

Let $G \leq sym(\Omega)$ be k-transitive for $k \geq 5$, and any subset Ω' of Ω with $|\Omega'| =$ k.Suppose that $H = G_{\Omega'}$ is the stabilizer of G, then $H = soc(G)$

Proof

The result follows from Theorem 3.1, if H is a subgroup such that $\mathrm{H} = \mathrm{soc(G)}$, then by Theorem 2.13, there exist a normal subgroup $\mathrm{K} \leq \mathrm{H}$ such that the normalizer $\mathrm{H} = \mathrm{N_G}(K)$ in Ω' is invariant which also follow from Lemma 2.11. Hence H acts k-transitively on Ω'. Further by Theorem 2.13, it shows that the socle of G is a subgroup of the normalizer of the group as in Theorem 3.1, Hence H acts k-homogeneously.

CONCLUSION

Based on the results ascertained, we can clearly states that, the of a group is unique or at most two in the case of primitive groups. Also the socle types is basically lying solely on the classification of the finite primitive groups as given in the preliminary results. Further, the degree of homogeneity based on the socle type showed that socle of a group were invariant. It was investigated in relation to the degree of homogeneity that socle is not necessarily transitively normal.

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