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THE PROPERTIES OF TOPP LEONE EXPONENTIATED INVERSE WEIBULL DISTRIBUTION WITH APPLICATION TO SURVIVAL ANALYSIS

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ABSTRACT: *In this study, a new four-parameter lifetime distribution called the Topp Leone exponentiated inverse Weibull distribution was introduced. The graphs of the probability distribution function, cumulative density function and hazard function are presented to ascertain the shapes of the distribution. It is seen from the graphs that the shape of the distribution is skewed, symmetric, reversed J-shape, increasing, decreasing and bathtub and unimodal. A linear representation for the probability distribution function was carried out. Some mathematical properties of the distribution were presented such as moments, moment generating function, quantile function, survival function, hazard function. The distribution of order statistic was obtained. Estimation of the parameters by maximum likelihood method was discussed. An application to real-life of the distribution was presented and the result showed the fit and flexibility of the new distribution over some lifetime distributions considered. The analysis showed that the model is effective in fitting survival data.*

KEYWORDS: Complementary Weibull, Generalized Inverse Weibull, Inverse Gamma, Real-Life, Survival Analysis, Topp Leone Exponentiated-G, Unimodal Failure Rate.

INTRODUCTION

Inverse distributions, namely, inverse Gamma, inverse generalized Gamma, inverse Weibull has been studied in literature (Pawlas and Szynal, 2000). Keller *et al*. (1982) studied shapes of density and failure rate function for the basic inverse model and Drapella (1993) worked on inverse Weibull (IW) distribution. Drapella (1993) and Mudholkar and Kolia (1994) suggested the names complementary Weibull and reciprocal Weibull. The inverse Weibull distribution has been applied to a wide range of situations including applications in medicine, reliability and ecology. Baharith *et al*., (2014) introduced an extension of the inverse Weibull distribution called the beta generalized inverse Weibull distribution. Gusmao *et al*., (2011) defined a three-parameter generalized inverse Weibull distribution with decreasing and unimodal failure rate. Khan and King 2012 proposed the modified inverse Weibull distribution and discussed its various properties. Khan (2010) introduced and studied a four-parameter distribution, so-called the beta inverse Weibull distribution and Shahbaz *et al*., (2012) introduced and studied a four-parameter inverse Weibull distribution. Rodrigues *et al*., (2016) introduced exponentiated Kumaraswamy inverse Weibull distribution. Pararai *et al*., (2014) introduced and studied gamma-inverse Weibull. Jain *et al*., (2014) worked on inverse generalized Weibull and generalized inverse generalized Weibull (GIGW) distributions. Mudasir and Ahmad (2018) proposed generalized inverse Weibull distribution. Basheer

(2019) proposed alpha power inverse Weibull distribution. Elbatal and Muhammed (2014) introduced exponentiated generalized inverse Weibull distribution.

A random variable X has an inverse Weibull distribution if its cumulative distribution function (cdf) is of the form

$$H(x;\lambda,\beta) = e^{-\left(\frac{\beta}{x}\right)^{\lambda}}, x > 0, \lambda, \beta > 0 \tag{1}$$

where λ is the shape parameter and β is the scale parameter. The probability density function corresponding to Eq. (1) is

$$h(x;\lambda,\beta) = \lambda\beta^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\beta}{x}\right)^{\lambda}} \tag{2}$$

Many statistical distributions have been developed which are very useful in describing and predicting real-world phenomena. Although, there are always rooms for developing distributions which are either more flexible or for fitting specific real-world scenarios. This has motivated many researchers in developing new and more flexible models. In developing these flexible distributions, researchers have proposed classes of distributions. Some recent classes of distribution are: Topp Leone odd Lindley-G family of distributions by Reyad *et al*., (2018), Topp Leone generated family of distributions by Aryuyuen (2018), Marshall-Olkin Odd Lindley-G family of distributions by Jamal (2019), Exponentiated Kumaraswamy-G family of distributions by Silva *et al*., (2019), Fréchet Topp Leone-G family of distributions by Reyad *et al*., (2019), Topp Leone generalized inverted Kumaraswamy distributions by Reyad *et al*., (2019), Power Lindley-G family of distributions by Hassan and Nassr (2019), Modi family of continuous probability distributions by Modi *et al*., (2020), Topp Leone Exponentiated-G family of distributions by Ibrahim *et al*., (2020).

The main objectives of this research are to obtain a more flexible distribution by adding two extra shape parameters from the Topp Leone exponentiated-G family proposed by Ibrahim *et al*., (2020) to the baseline distribution and to improve goodness-of-fit to real-life data.

The rest of the paper is organized as follows. In Section 2 the Topp Leone exponentiated inverse Weibull distribution is defined and some special and the graphs of its pdf and cdf are presented. Infinite mixture representation of the pdf of Topp exponentiated inverse Weibull is studied in section 3. Some statistical and mathematical properties of the Topp Leone exponentiated inverse Weibull distribution are explored in Section 4. The estimation of the model parameters using the methods of moments and maximum likelihood is discussed in Section 5. Distribution of order statistic is presented in section 6. In Section 7, an application to a real data set is reported. Section 8 presented the concluding remarks and finally, in section 9 future researches is discussed.

THE NEW MODEL

Ibrahim *et al*., (2020) proposed a family of continuous distributions known as Topp Leone exponentiated-G (TLEx-G) family. Let $H(x;\,\varphi)$ be the cdf of a random variable X. The cdf of TLEx-G is given by

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$$F(x;\alpha,\theta,\varphi)=\{1-[1-H(x;\ \varphi)^{\alpha}]^2\}^{\theta} \tag{3}$$

and the pdf corresponding to Eq. (3) is

$$f(x;\alpha,\theta,\varphi)=2\alpha\theta h(x;\ \varphi)H(x;\ \varphi)^{\alpha-1}[1-H(x;\ \varphi)^{\alpha}]\{1-[1-H(x;\ \varphi)^{\alpha}]^2\}^{\theta-1} \tag{4}$$

$$x>0,\qquad \alpha,\qquad \theta,\qquad \varphi>0$$

Where $h(x;\varphi)$ is the baseline pdf, α and θ are positive shape parameters and φ is the vector of parameters.

Then, the cdf of Topp Leone exponentiated inverse Weibull (TLExIW) distribution is

$$F(x;\alpha,\theta,\lambda,\beta)=\left\{1-\left[1-\left(e^{-\left(\frac{\beta}{x}\right)^{\lambda}}\right)^{\alpha}\right]^2\right\}^{\theta} \tag{5}$$

The pdf corresponding to Eq. (5) is

$$f(x;\alpha,\theta,\lambda,\beta)=2\alpha\theta\lambda\beta^{\lambda}x^{-(\lambda+1)}\left(e^{-\left(\frac{\beta}{x}\right)^{\lambda}}\right)^{\alpha}\left[1-\left(e^{-\left(\frac{\beta}{x}\right)^{\lambda}}\right)^{\alpha}\right]\left\{1-\left[1-\left(e^{-\left(\frac{\beta}{x}\right)^{\lambda}}\right)^{\alpha}\right]^2\right\}^{\theta-1} \tag{6}$$

$$x>0,\alpha,\theta,\lambda,\beta>0.$$

Where β is the scale parameter and α,θ,λ are the positive shape parameters of the new distribution.

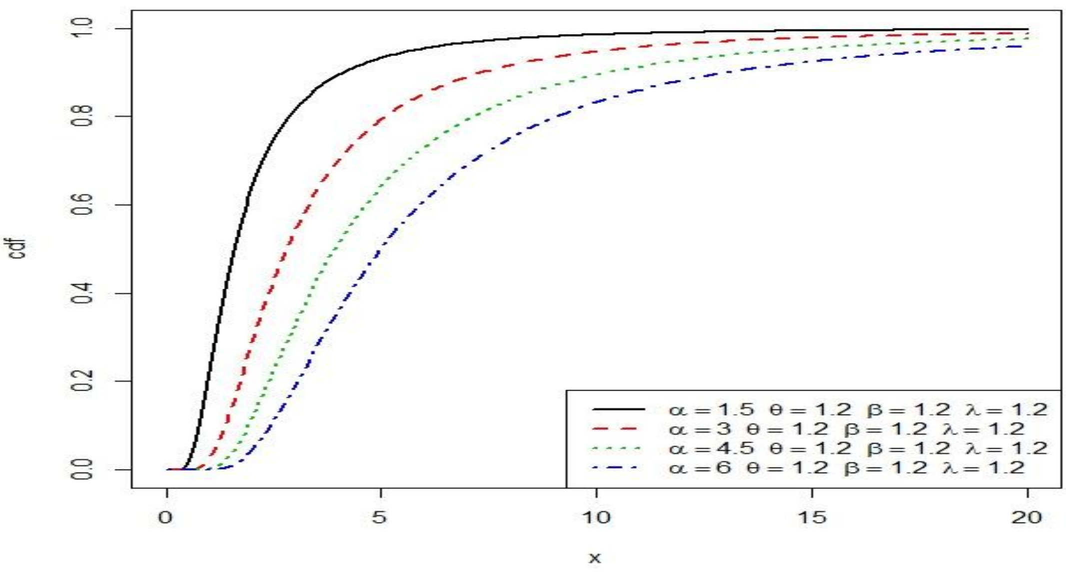

Figure 1: CDF Plot of TLExIW Distribution with Different Values of α and Parameters β,θ,λ Fixed at 1.2

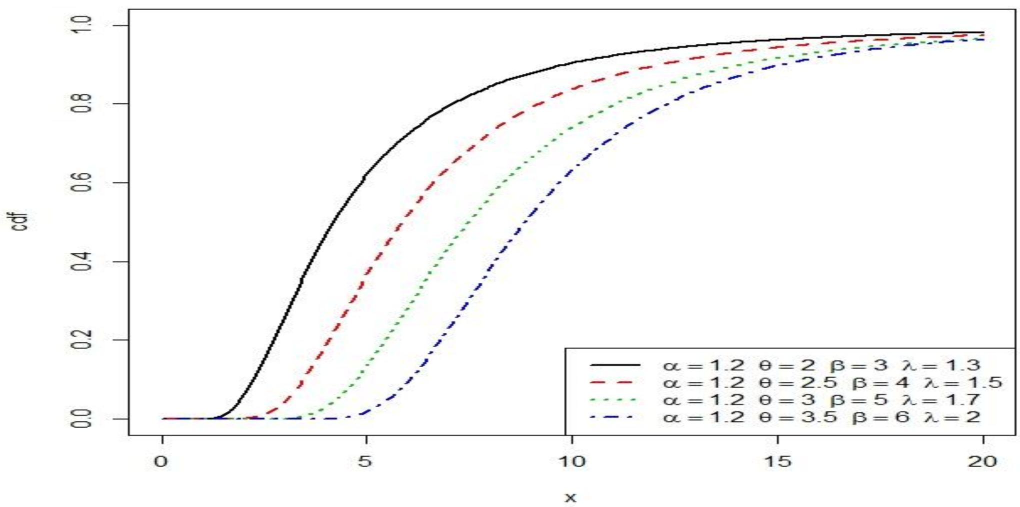

Figure 2: CDF Plot of TLExIW Distribution with Values of α Fixed at 1.2 and Parameters β, θ, λ varies

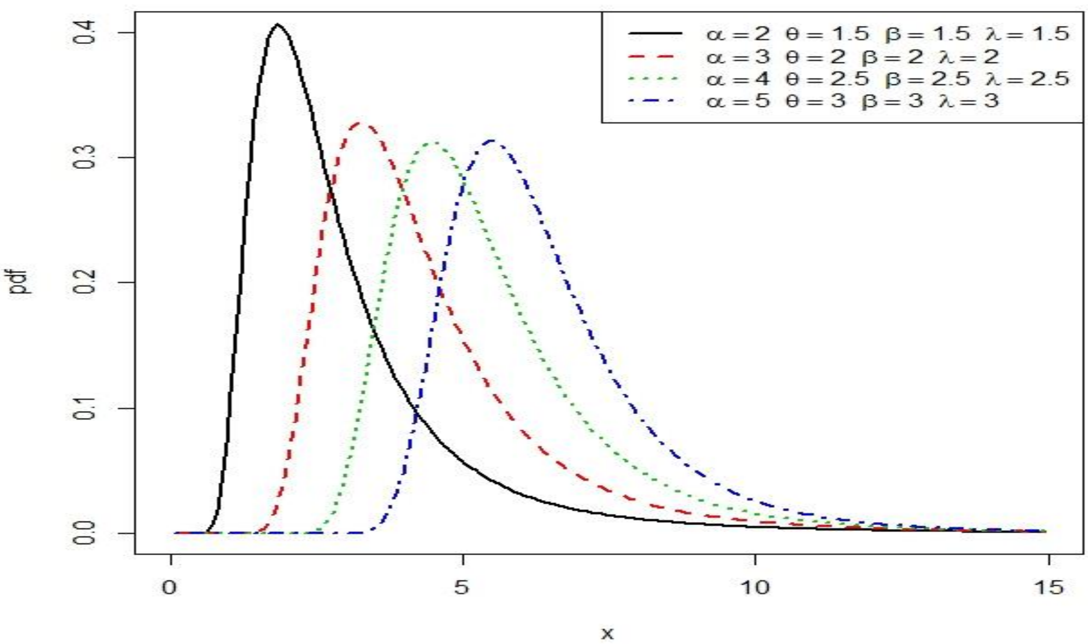

Figure 3: Pdf Plot of TLExIW Distribution with Different Parameter Values

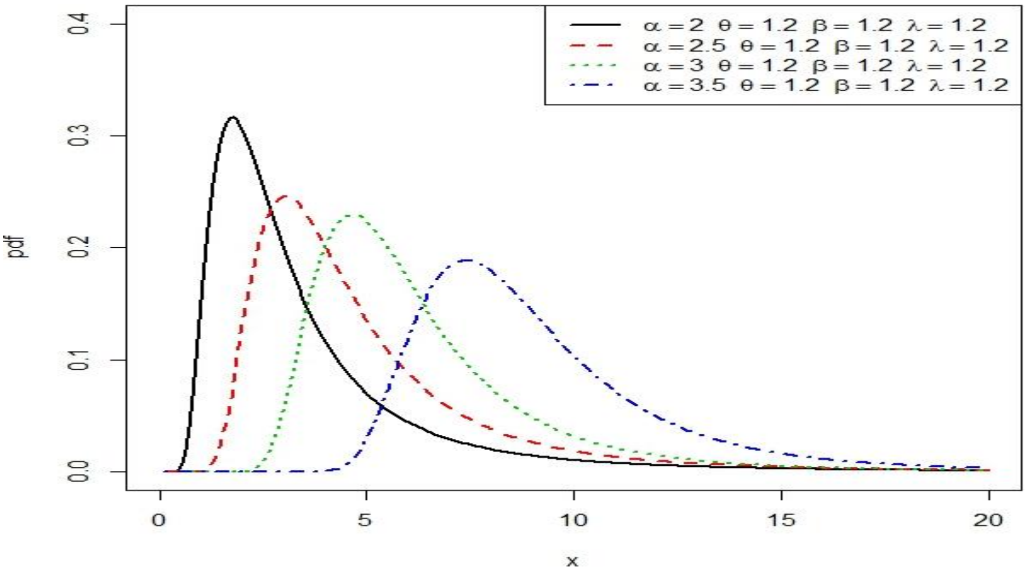

Figure 4: PDF plot of TLExIW Distribution with Different Values of α and Parameters θ, β, λ Fixed at 1.2

INFINITE MIXTURE REPRESENTATION

Using the binomial expansion

$$(1-x)^{\theta} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\theta)}{i! \Gamma(\theta-i)} x^i \tag{7}$$

on pdf in Eq. (6), we have

$$f(x;\alpha,\theta,\beta,\lambda) = 2\alpha\theta\lambda\beta^{\lambda} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j} \Gamma(\theta)\Gamma(2(i+1))}{i! j! \Gamma(\theta-i)\Gamma(2(i+1)-j)} x^{-(\lambda+1)} \left(e^{-\left(\frac{\beta}{x}\right)^{\lambda}}\right)^{\alpha(j+1)} \tag{8}$$

Equation (8) is the linear representation of Eq. (6).

STATISTICAL AND MATHEMATICAL PROPERTIES

Moments

$$E(X^r) = \int_0^{\infty} x^r f(x) dx \tag{9}$$

$$E(X^r) = 2\alpha\theta\beta^r \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j} \Gamma(\theta)\Gamma(2(i+1))\Gamma(1-\frac{r}{\lambda})}{i! j! \Gamma(\theta-i)\Gamma(2(i+1)-j)(\alpha(j+1))^{1-\frac{r}{\lambda}}} \tag{10}$$

The mean of TLExIW distribution is obtained by setting $r = 1$ in Eq. (1) to get

$$E(X) = 2\alpha\theta\beta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j} \Gamma(\theta)\Gamma(2(i+1))\Gamma(1-\frac{1}{\lambda})}{i! j! \Gamma(\theta-i)\Gamma(2(i+1)-j)(\alpha(j+1))^{1-\frac{1}{\lambda}}} \tag{11}$$

Moment Generating Function

$$E(e^{tx}) = \int_0^\infty x^r f(x)dx \tag{12}$$

$$e^{tx} = \sum_{m=0}^{\infty} \frac{t^m x^m}{m!} \tag{13}$$

$$E(e^{tx}) = 2\alpha\theta\beta^m \sum_{i=o}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j}\Gamma(\theta)\Gamma(2(i+1))\Gamma(1-\frac{m}{\lambda})t^m}{i!j!m!\Gamma(\theta-i)\Gamma(2(i+1)-j)(\alpha(j+1))^{1-\frac{m}{\lambda}}} \tag{14}$$

Quantile function

If u has a uniform $U(0,1)$ distribution, the quantile function can be obtained by:

$$Q(u) = F^{-1}(u) \tag{15}$$

Inverting the cdf in Eq. (5), we obtained the quantile function of TLExIW distribution as

$$Q(u) = x = -\beta\left\{log\left(\left[1-\left[1-u^{\frac{1}{\theta}}\right]^{\frac{1}{2}}\right]^{\frac{1}{\alpha}}\right)\right\}^{\frac{-1}{\lambda}} \tag{16}$$

The median of the TLExIW distribution can be obtained by setting $u = 0.5$ in Eq. (16)

$$Q(0.5) = x_m = -\beta\left\{log\left(\left[1-\left[1-0.5^{\frac{1}{\theta}}\right]^{\frac{1}{2}}\right]^{\frac{1}{\alpha}}\right)\right\}^{\frac{-1}{\lambda}} \tag{17}$$

Survival function

The survival function is also known as reliability function, is the probability of an item not failing prior to some time. It can be defined as

$$S(x) = 1 - F(x) \tag{18}$$

$$S(x) = 1 - \left\{1-\left[1-\left(e^{-\left(\frac{\beta}{x}\right)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta} \tag{19}$$

Hazard rate function

The hazard rate function is given as

$$\tau(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} \tag{20}$$

$$\tau(x) = \frac{2\alpha\theta\lambda\beta^{\lambda}x^{-(\lambda+1)}\left(e^{-\left(\frac{\beta}{x}\right)^{\lambda}}\right)^{\alpha}\left[1-\left(e^{-\left(\frac{\beta}{x}\right)^{\lambda}}\right)^{\alpha}\right]\left\{1-\left[1-\left(e^{-\left(\frac{\beta}{x}\right)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta-1}}{1-\left\{1-\left[1-\left(e^{-\left(\frac{\beta}{x}\right)^{\lambda}}\right)^{\alpha}\right]^{2}\right\}^{\theta}} \tag{21}$$

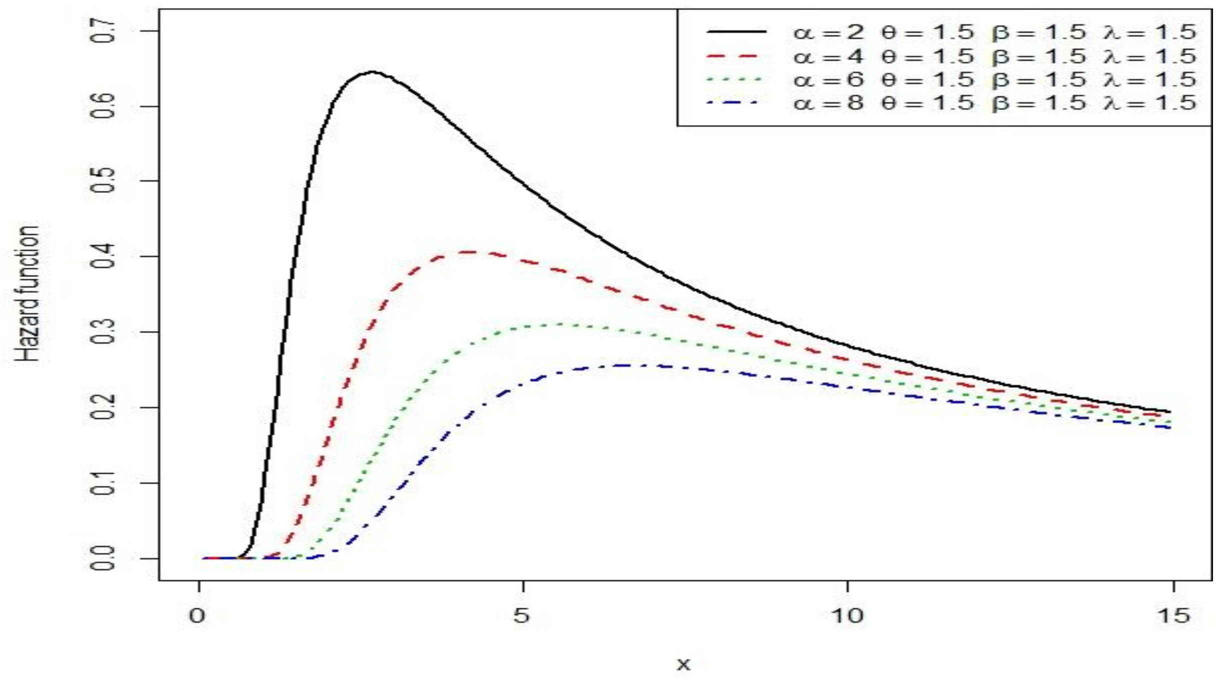

Figure 5: Hazard Function Plot for TLExIW with Parameter α Varies and Parameters β, θ, λ Fixed at 1.5

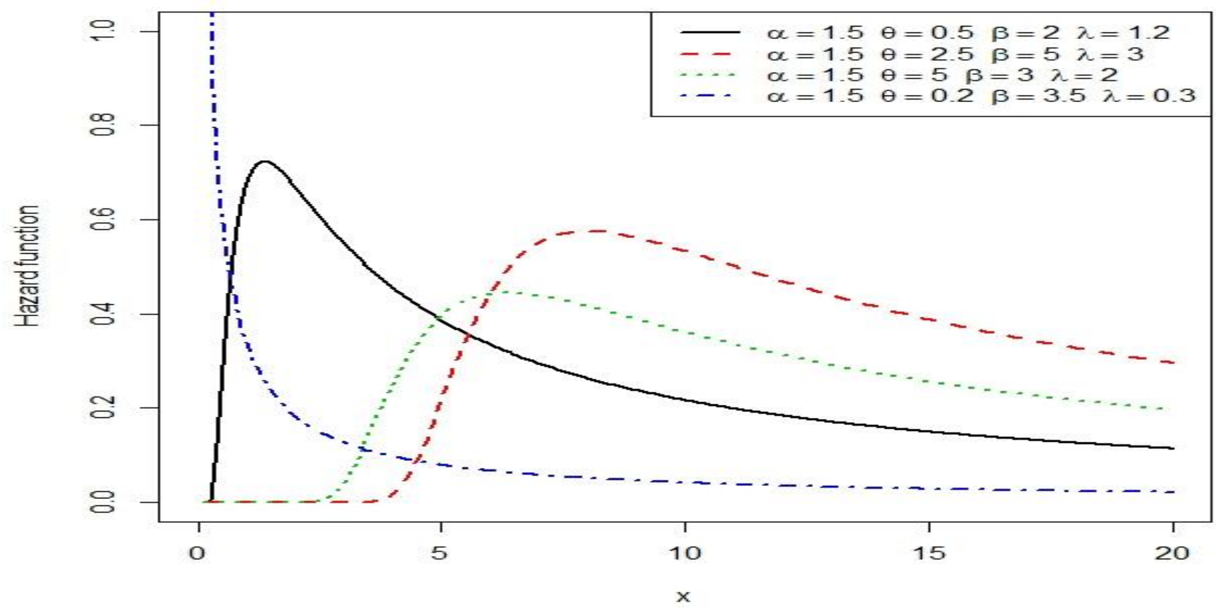

Figure 6: Hazard Function Plot for TLExIW with Parameter α Fixed at 1.5 and Parameters β, θ, λ Varies

ESTIMATION

In this section the method of maximum likelihood estimation is used to estimate the parameters of the TLExIW distribution. Let $X_{1,}X_2, \dots, X_n$ be independent and identically distributed (iid) observed random sample of size n from the TLExIW distribution. Then, the log-likelihood function based on observed sample for the vector of parameter $\emptyset = (\alpha, \beta, \theta, \lambda)^T$ is given by

$$logL = nlog2 + nlog\alpha + nlog\theta + nlog\lambda + n\lambda log\beta - (\lambda+1)\sum_{i=1}^{n} log(x_i) - \alpha\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\lambda} + \sum_{i=1}^{n} log\left[1-\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}\right] + (\theta-1)\sum_{i=1}^{n} log\left\{1-\left[1-\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}\right]^2\right\} \quad (22)$$

The components of score vector $U = (U_\alpha, U_\beta, U_\theta, U_\lambda)^T$ are given as

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\lambda} - \sum_{i=1}^{n}\frac{\left(\frac{\beta}{x_i}\right)^{\lambda}}{1-\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}} + (\theta-1)\sum_{i=1}^{n}\frac{2\left(\frac{\beta}{x_i}\right)^{\lambda}\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}\left[1-\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}\right]}{1-\left[1-\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}\right]^2} \quad (23)$$

$$U_\beta = \frac{n\lambda}{\beta} - \frac{\alpha}{\lambda}\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\lambda} - \sum_{i=1}^{n}\frac{\alpha\lambda\left(\frac{\beta}{x_i}\right)^{\lambda-1}\left(\frac{1}{x_i}\right)}{1-\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}} + (\theta 1)\sum_{i=1}^{n}\frac{2\alpha\lambda\beta^{-1}\left(\frac{\beta}{x_i}\right)^{\lambda}\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}\left[1-\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}\right]}{1-\left[1-\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}\right]^2} \quad (24)$$

$$U_\lambda = \frac{n}{\lambda} + nlog\beta - \sum_{i=1}^{n} log(x_i) + \alpha\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\lambda} log\left(\frac{\beta}{x_i}\right) + \sum_{i=1}^{n}\frac{\alpha\left(\frac{\beta}{x_i}\right)^{\lambda} log\left(\frac{\beta}{x_i}\right)}{1-\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}} - (\theta-1)\sum_{i=1}^{n}\frac{2\alpha\left(\frac{\beta}{x_i}\right)^{\lambda} log\left(\frac{\beta}{x_i}\right)\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}\left[1-\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}\right]}{1-\left[1-\left(e^{-\left(\frac{\beta}{x_i}\right)^{\lambda}}\right)^{\alpha}\right]^2} \quad (25)$$

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n} log\left\{1 - \left[1 - \left(e^{-\left(\frac{\beta}{x_i}\right)^\lambda}\right)^\alpha\right]^2\right\} \tag{26}$$

The solution of the non-linear equations (23), (24), (25) and (26) cannot be obtained analytically so, we resolved to use iterative methods.

ORDER STATISTIC

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be n independent random variable from TLExIW distribution and let $X_{(1)} \leq X_{(2)} \leq, \dots, \leq X_{(n)}$ be their corresponding order statistic.

Let $F_{r:n}(x)$ and $f_{r:n}(x)$, $r = 1, 2, 3, \dots, n$ denote the cdf and pdf of the r^{th} order statistic $X_{r:n}$ respectively. The pdf of $X_{r:n}$ is given as

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)} f(x)[F(x)]^{r-1}[1 - F(x)]^{n-r} \tag{27}$$

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)} \sum_{i=0}^{n-r} (-1)^i f(x)[F(x)]^{r+i-1} \tag{28}$$

Inserting Eq. (5) and Eq. (6) into Eq. (28), we have

$$f_{r:n}(x) = \frac{2\alpha\theta\lambda\beta^\lambda x^{-(\lambda+1)}}{B(r,n-r+1)} \sum_{i=0}^{n-r} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(\theta(r+1))\Gamma(2(j+1))}{j!k!\Gamma(\theta(r+1)-j)\Gamma(2(j+1)-k)} \left[e^{-\left(\frac{\beta}{x}\right)^\lambda}\right]^{\alpha(k+1)} \tag{29}$$

Equation (29) is the r^{th} order statistic of the TLExIW distribution.

Therefore, the pdf of the minimum and maximum order statistic of the TLExIW distribution are obtained by setting $r = 1$ and $r = n$ respectively in Eq. (29).

APPLICATION TO REAL-LIFE

In this section, an application of the TLExIW distribution to real-life data set to demonstrate the flexibility of the distribution is presented. The data is fitted to the TLExIW distribution and other distributions such as the inverse Weibull, Exponentiated Weibull and Weibull distributions.

- Exponentiated Weibull (ExW) distribution

$$f(x; \alpha, \beta, \lambda) = \alpha\lambda\beta^\lambda x^{\lambda-1} e^{-(\beta x)^\lambda}\left[1 - e^{-(\beta x)^\lambda}\right]^{\alpha-1}$$

- Weibull (W) distribution

$$f(x; \lambda, \beta) = \lambda\beta^\lambda x^{\lambda-1} e^{-(\beta x)^\lambda}$$

- Inverse Weibull (IW) distribution

$$f(x;\lambda,\beta)=\lambda\beta^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\beta}{x}\right)^{\lambda}}$$

- Inverse exponential (IEx) distribution

$$f(x;\ \beta)=\left(\frac{\beta}{x^{2}}\right)e^{-\left(\frac{\beta}{x}\right)}$$

The data set from Bjerkedal (1960) represents the survival times, in days of guinea pigs injected with different doses of tubercle bacilli. The data has been used by Pararai et al (2014). The data set consists of 72 observations and are listed below:

12, 15, 22, 24 ,24, 32, 32,33, 34, 38, 38, 43,44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68,70, 70, 72, 73, 75, 76,76, 81, 83, 84, 85, 87, 91, 95,96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258,258, 263, 297, 341, 341, 376

Table 1: The ML Estimate of the Parameters and AIC Value.

Distributions	Estimates	-LL	AIC
TLExIW	$\hat{\alpha}=14.4682$ $\hat{\beta}=5.0757$ $\hat{\theta}=1.3721$ $\hat{\lambda}=1.0180$	392.3285	792.6571
IW	$\hat{\beta}=54.2009$ $\hat{\lambda}=1.4146$	395.6491	795.6491
W	$\hat{\beta}=0.0090$ $\hat{\lambda}=1.3931$	397.1477	798.2953
ExW	$\hat{\alpha}=0.7725$ $\hat{\beta}=0.0070$ $\hat{\lambda}=0.6207$	403.2675	812.5349
IEx	$\hat{\beta}=60.0641$	402.6718	807.3437

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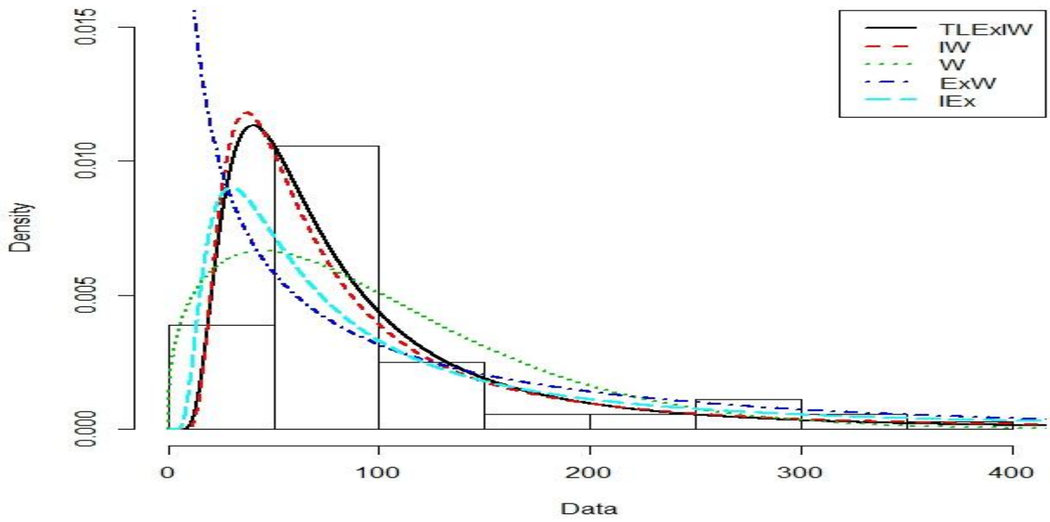

CONCLUSION

In this article, we derived and studied a new continuous probability distribution called the Topp Leone Exponentiated Inverse Weibull distribution using the family of distributions proposed by Ibrahim et al. (2020). Some of its mathematical properties comprising the moments, moment generating function, quantile function, order statistics and reliability analysis were derived. The parameters of the distribution were also obtained using the method of maximum likelihood. A real-life application to survival times, in days of guinea pigs injected with different doses of tubercle bacilli was carried out to test the fitness of the new distribution with some other known distributions such as IW, W, ExW and IEx distributions. The result as shown in Table 1 showed that the new distribution which is the Topp Leone exponentiated inverse Weibull distribution is the best distribution for fitting the data set among the distributions considered in this article. Also, it can be seen from figure 7 that the new distribution fits the data better compared to the distributions considered.

Future Research

Future research can be carried out in using different estimation methods to estimate the parameters of the distribution and compare with the maximum likelihood method used in this current research. The distribution can also be applied in other areas to check for its flexibility.

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