African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 3, Issue 4, 2020 (pp. 1-11)
www.abjournals.org

MATHEMATICAL ANALYSIS OF BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION ON STOCK PRICES

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ABSTRACT: *This paper is centered on Black-Scholes (BS) partial differential equation (PDE) and Crank-Nicolson (CN) numerical solutions for the valuation of European call option. In particular, Crank-Nicolson's method was successfully applied to the transformed boundary value problems of European call option with results obtained. It was observed that Black-Scholes and Crank-Nicolson are impossible to differentiate but in terms of precisions Black-Scholes analytical values were found to be adequate. Also, a statistical test was performed, using Kolmogorov-Smirnov (KS) for goodness of fit test; results showed that BS and CN came from a common distribution. However, this paper presented here has a vital role in this dynamic area of financial Mathematics.*

KEYWORDS: Crank-Nicolson, Call Option, Kolmogorov-Smirnov, Stock Prices, Modeling.

INTRODUCTION

In financial Mathematics, the dynamics of financial markets cannot be over emphasized as Black-Scholes (BS) Partial Differential Equation (PDE) explores the changes of option (values) as a function of security and time. In the course of this PDE the analytical and Crank-Nicolson numerical solutions were established. In the work of [1] they obtained a mathematical structure for finding the realistic price of European options by the use of no-arbitrage principle to describe a PDE which governs the growth of the option price that evolves time to expiration. The facts of this paper can be seen in the following books: [3], [5],[6],[7],[8],[9] and [13] etc. This paper deals with numerical solution to famous Black-Scholes partial differential equation which has a closed form solution of European options; because of its numerous applications researchers has used numerical methods under different circumstances to estimate option prices such as [2][11],[12],[13],[14],[15] and [16] etc. The option under deliberation in this study is European call option. In other words, an option on underlying asset is a business between parties who come together to agree on either buying or selling an underlying asset at a determined strike price in the future for a fixed price. The holder has the right but cannot be compelled to buy, for call option where European put option involves the ability to sell an asset for a certain charge at a prescribed date in the future.

The aim of this paper is to compare analytically, numerically and to identify the sources of relative errors between Black-Scholes and Crank-Nicolson as errors in valuation can cause unreasonable loss for trader [2], hence using Kolmogorov-Smirnov (KS) for goodness of fit test. The KS statistic belongs to nonparametric test in statistics. It is a very useful test for

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www.abjournals.org

testing normality assumption and based on the largest vertical difference between the hypothesized and empirical distribution, besides its use in the test for assumptions of normality, it is popular in the test to verify if two independent samples are drawn from a common distribution.

METHODOLOGY

The seminar paper on option pricing by [1],[4] has had great impact on contemporary research in Mathematics of finance. This model is commonly used in financial modeling. The Black-Scholes model is made up of on seven assumptions:

- The asset price has characteristics of a Brownian motion with μ and σ as constants.
- The transaction costs or taxes are not allowed.
- The entire securities are absolutely divisible.
- Dividend is not permitted during the period of the derivatives.
- Unacceptable of riskless arbitrage opportunities.
- The security trading is constant.
- The option is exercised at the time of expiry for both call and put options.

The analytic formula for the prices of European call option is given as follows

$$\left.\begin{aligned} C &= SN(d_1) - Ke^{-rt}N(d_2) \\ d_1 &= \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}\right\} \tag{1}$$

where C is Price of a call option, S is price of underlying asset, K is the strike price, r is the riskless rate, T is time to maturity, σ^2 is variance of underlying asset, σ is standard deviation of the (generally referred to as volatility) underlying asset, and N is the cumulative normal distribution.

Derivation of black-Scholes (BS) partial differential equation (PDE)

According to [3], [12], the derivation of BS PDE is based on Itô process with an assumption that the stock prices follow a geometric Brownian motion, ie

$$dS = \mu S dt + \sigma S dx \tag{2}$$

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ISSN: 2689-5323
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where S is the stock price, μ is the drift, σ is the volatility of underlying asset and dx is a Wiener process.

Theorem 1 (Black-Scholes Equation).Let the value of an option be $V(t,S)$,standard deviation of stock be stock's returns be σ ,and risk-free interest rate be r. Then the price of an option over time can be expressed by the following partial differential equation:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

Proof

Suppose we have an option whose worth $V(S,t)$ depends only on S and t . Assume also that the asset price is varied by a small amount of change dS , then the function V will also be affected. By means of Itô lemma

$$dV=\sigma S\frac{\partial V}{\partial S}dx+\left(\mu S\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\frac{\partial V}{\partial t}\right)dt \tag{3}$$

According to [14], the value of one portfolio having one stock can be expressed with the function $V(S,t)$.

$$\pi = V-\Delta S \tag{4}$$

The change in the portfolio at time dt in (4) is given by

$$d\pi = dV-\Delta S \tag{5}$$

Putting (1), (2) into (4), we find that π follows random walk given by.

$$d\pi=\sigma S\left(\frac{\partial V}{\partial S}-\Delta\right)dx+\left(\mu S\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\frac{\partial V}{\partial t}-\mu\Delta S\right)dt$$

To eliminate the random component in this random walk, let

$$\Delta=\frac{\partial V}{\partial S} \tag{6}$$

Note that Δ is the value of $\frac{\partial V}{\partial S}$ at the start of the time step dt . This results in a portfolio whose increment is wholly deterministic so that

$$d\pi=\left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt \tag{7}$$

Now that the portfolio is riskless it should earn riskless return. The change in the portfolio at time dt becomes (after substituting (1) and (2) into (4) and dividing through by dt)

$$d\pi = r\pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt,$$

and

$$r\left(V - \Delta S\right)dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt$$

which implies that

$$rV - rS\frac{\partial V}{\partial S} = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

This gives the solution

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{8}$$

This is the Black-Scholes partial differential equation as established.

European Call Option

The BS PDE for European Call Option with value $C(S,t)$ is given in the following equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{9}$$

The following initial and boundary conditions for Call Option:

$$\left.\begin{array}{r} C(0,t) = 0 \\ C(S,t) \quad\quad = S \;\; when\; S \to \infty \\ C(S,T) = \max(S-K,0) \end{array}\right\} \tag{10}$$

The Crank-Nicolson Finite Difference Method for an Option Pricing Model

The Crank-Nicolson finite difference method is to conquer the stability short-comings by applying the stability and convergence restrictions of the explicit finite difference methods. It is essentially an average of the implicit and explicit methods. However, to implement Crank-Nicolson approximation scheme on Black-Scholes partial differential equation, there must be a price time mesh in order to enhance efficiency as solution exits, the vertical axis in the mesh denotes the stock prices, while the horizontal axis denotes time. Therefore, every grid point in the mesh denotes a horizontal index i and a vertical index j such that every point in the mesh is the option price for a distinct time and a distinct stock price. At every time in

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the mesh $j\Delta s$ is equivalent to the stock price, and $i\Delta t$ is equivalent to the time. There exist boundary conditions which help in the numerical calculations; by means of the pay-off function. The maturity period, $t = T$ and the option are well computed for all the different initial stock prices using a boundary conditions for uniqueness of solution. To get the prices at $t = 0$, the model solves backwards for every time step from $t = T$, [15].

Recall that the Black-Scholes partial differential equation (9).

Let a function $V(S,t)$ in two-dimensional grid points, that is to say i and j stands for the index for stock price, S and time, t respectively. The function $V(S,t) = V_i^j$ can be stated as follows in the subsequent difference scheme [4],[11].

$$Z_i^j = \frac{1}{2}\sigma^2 S^2 DSS + rS_i DS - rV_i^j \tag{11}$$

Where DS is the first partial derivative and DSS is the second partial derivative of equation (3.5.6).

$S = i\Delta S$, for $0 \le i \le m$, $t = j\Delta t$ for $0 \le j \le i$

and

$$DSS = \frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{\Delta^2} \tag{12}$$

$$DS = \frac{V_{i+j}^j - V_{i-1}^j}{2\Delta S} \tag{13}$$

Taking forward difference and backward difference approximations respectively yields implicit and explicit schemes given below.

If we use a forward difference approximation to the time partial derivative, we obtain explicit scheme

$$\frac{V_i^{j+1} - V_i^j}{\Delta t} + Z_i^j = 0 \tag{14}$$

and similarly, we obtain the implicit scheme

$$\frac{V_i^{j+1} - V_i^j}{\Delta t} + Z_i^{j+1} = 0 \tag{15}$$

The averages of equations (14) and (15) yields Crank-Nicolson method of approximation

$$\frac{V_i^{j+1} - V_i^j}{\Delta t} + \frac{1}{2}\left(Z_i^j + Z_i^{j+1}\right) = 0 \tag{16}$$

From equation (16)

$$V_i^j - \frac{\Delta t}{2} Z_i^j = V_i^{j+1} + \frac{\Delta t}{2} Z_i^{j+1} \tag{17}$$

where

$$Z_i^j = \frac{\sigma^2 S_i^2}{2(\Delta S)^2}\left[V_{i+1}^j - 2V_i^j + V_{i-1}^j\right] + \frac{rS_i}{2\Delta S}\left[V_{i+1}^j - V_{i-1}^j\right] - rV_i^j$$

and

$$Z_i^{j+1} = \frac{\sigma^2 S_i^2}{2(\Delta S)^2}\left[V_{i+1}^{j+1} - 2V_i^{j+1} + V_{i-1}^{j+1}\right] + \frac{rS_i}{2\Delta S}\left[V_{i+1}^{j+1} - V_{i-1}^{j+1}\right] - rV_i^{j+1}$$

Collecting like terms of V_{i-1}, V_i *and* V_{i+1} and simplifying gives

$$Z_i^j = V_{i-1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta tS_i}{2\Delta S}\right] + V_i^j\left[\frac{2\Delta t}{\Delta t} + \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right] - V_{i+1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta tS_i}{2\Delta S}\right] \tag{18}$$

and

$$Z_i^{j+1} = V_{i-1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta tS_i}{2\Delta S}\right] + V_i^{j+1}\left[\frac{2\Delta t}{\Delta t} - \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t}{2}\right] + V_{i+1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta tS_i}{2\Delta S}\right] \tag{19}$$

Using (18) and (19) solving simultaneously and taking the average of these two equations we obtain

$$\left.\begin{aligned} &V_{i-1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} - \frac{r\Delta tS_i}{4\Delta S}\right] + V_i^j\left[1 + \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right] - V_{i+1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} + \frac{r\Delta tS_i}{4\Delta S}\right] \\ &= V_{i-1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} - \frac{r\Delta tS_i}{4\Delta S}\right] + V_i^{j+1}\left[1 - \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t}{2}\right] + V_{i+1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} + \frac{r\Delta tS_i}{4\Delta S}\right] \end{aligned}\right\} \tag{20}$$

The expressions inside the square brackets of (20) are replaced with the a_i, b_i, c_i. We obtain the following equation

$$a_i V_{i-1}^j + (1 + b_i)V_i^j - c_i V_{i+1}^j = a_i V_{i-1}^{j+1} + (1 - b_i)V_i^{j+1} + c_i V_{i+1}^{j+1} \tag{21}$$

where

$$a_i = \frac{\Delta t}{4}\left(\sigma^2 S_i^2 - rS_i\right), b_i = -\frac{\Delta t}{2}\left(\sigma^2 S_i^2 - r\right) \text{ and } c_i = \frac{\Delta t}{4}\left(\sigma^2 S_i^2 + rS_i\right).$$

a_i, b_i, c_i are random variables; $i = 0, 1, \ldots, M$.

The CN discretization out come in tridiagonal matrix which is solvable at each time step.

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 3, Issue 4, 2020 (pp. 1-11) www.abjournals.org

RESULTS AND DISCUSSION

Table 1: Comparing the performance of the Black-Scholes exact values and Crank-Nicolson finite difference method for European Call Option with different initial stock prices.

K = 25 T=1 r = 0.03			k=25 r = 0.05		Mean values of BS and CN	STD values of BS and CN	Min values
Sigma	So	BS	CN	RE			
0.2	30	6.1368	6.1325	0.0007	6.1347	0.0030	6.1325:CN
	35	10.8152	10.8148	0.0034	10.8150	0.000283	10.8148:CN
	40	15.7513	15.7515	0.000095	15.7514	0.014142	15.7513:BS
	45	20.7407	20.7409	0.0000096	20.7408	0.0001414	20.7407:BS
	50	25.7391	25.7399	0.000031	25.7395	0.000566	25.7391:BS
	55	30.7389	30.7423	0.00011	30.7406	0.0024	30.7389:BS
	60	35.7389	35.75	0.00031	35.7445	0.0078	35.7389:BS
	65	40.7389	40.7663	0.00067	40.7526	0.0194	40.7389:BS
	70	45.7389	45.79	0.00112	45.7644	0.0361	45.7389:BS
	75	50.7389	50.808	0.00136	50.7734	0.0489	50.7389:BS
0.25	30	6.478	6.4736	0.00068	6.4758	0.0031	6.4736:CN
	35	10.9682	10.967	0.000109	10.9676	0.08485	10.967:CN
	40	15.8046	15.8048	0.000013	15.8047	0.01442	10.967:CN
	45	20.7568	20.7585	0.000082	20.7576	0.0012	20.7568:BS
	50	25.7437	25.7492	0.00021	25.7464	0.0039	25.7437:BS
	55	30.7401	30.7547	0.00047	30.7474	0.0103	25.7437:BS
	60	35.7392	35.7696	0.00085	35.7544	0.0215	30.7401:BS
	65	40.739	40.7894	0.00124	40.7642	0.0356	40.739:BS
	70	45.7389	45.8034	0.00141	45.7712	0.0456	45.7389:BS
	75	50.7389	50.7922	0.00105	50.7656	0.0377	50.7389:BS

The Values in Table 1 were generated by putting in r=0.2, k=25 while allowing S_0 and σ vary in (1) and (20) for call options such that $s_0 = 30-75$, sigma=0.2 and 0.25 respectively presented in column 1 are the values of sigma, columns 3 and 4 are the exact values of BS and CN respectively. The 5th column gives the relative error (RE) of the difference between the estimated prices using BS and CN pricing schemes given by $|BS-CN|/BS$. RE is the ratio of absolute difference between BS and CN to BS such that when this ratio is very small, the performances of both BS and CN are equivalent, as can be observed in Table 1.A little increase of sigma also increases the option prices. Visual inspection shows that BS and CN are indistinguishable. The 6th and 7th columns show the means and standard deviations of both BS and CN. BS exact values were discovered to be best in terms of precisions; see the 8th column.

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ISSN: 2689-5323
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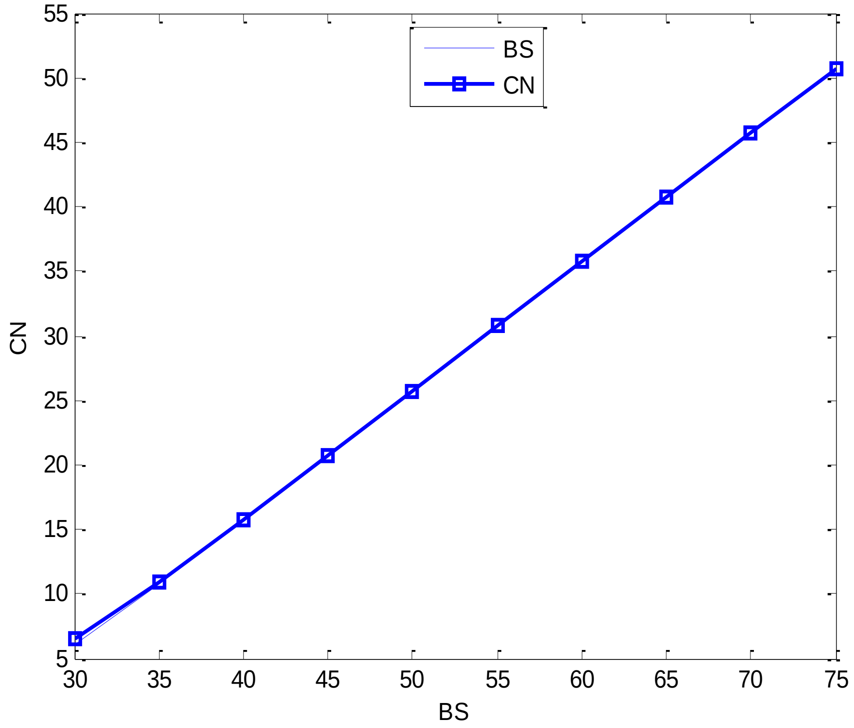

Figure 1: Graphical Representation of BS and CN

Figure 1 show a straight-line graph which goes to mean that Black-Scholes and Crank-Nicolson are identical. This agrees with our results in Table 1.

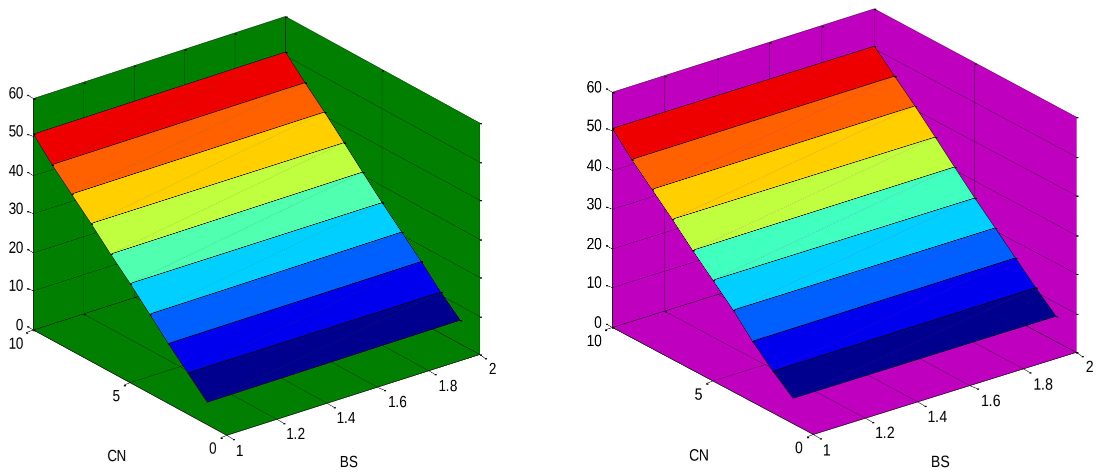

Figure 2: Mesh Surface Views of Black-Scholes and Crank-Nicolson Option Prices with Sigma 0.2 and 0.25 Respectively.

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ISSN: 2689-5323
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In Figure 2, the both plots are skewed to the left with constant increase throughout the trading days which might results to millions of naira etc. The plots portray uniformity between Black-Scholes and Crank-Nicolson which is fundamental to this paper.

Kolmogorov-Smirnov Test

H_0: The BS call option and CN numerical solutions are from the same distribution.

H_1: They are not from the same distribution.

Table 2: Goodness –of- Fit Test

Sigma	So	BS	CN	Results
0.2	30	6.1368	6.1325	Reject
	35	10.8152	10.8148	Reject
	40	15.7513	15.7515	Reject
	45	20.7407	20.7409	Reject
	50	25.7391	25.7399	Reject
	55	30.7389	30.7423	Reject
	60	35.7389	35.75	Reject
	65	40.7389	40.7663	Reject
	70	45.7389	45.79	Reject
	75	50.7389	50.808	Reject
0.25	30	6.478	6.4736	Reject
	35	10.9682	10.967	Reject
	40	15.8046	15.8048	Reject
	45	20.7568	20.7585	Reject
	50	25.7437	25.7492	Reject
	55	30.7401	30.7547	Reject
	60	35.7392	35.7696	Reject
	65	40.739	40.7894	Reject
	70	45.7389	45.8034	Reject
	75	50.7389	50.7922	Reject

Results displayed in Table 2, shows that at $\alpha =$ 0.01 and 0.05 respectively with different initial stock prices shows reject all through. This implies that there is no significant difference between Black-Scholes and Crank-Nicolson.

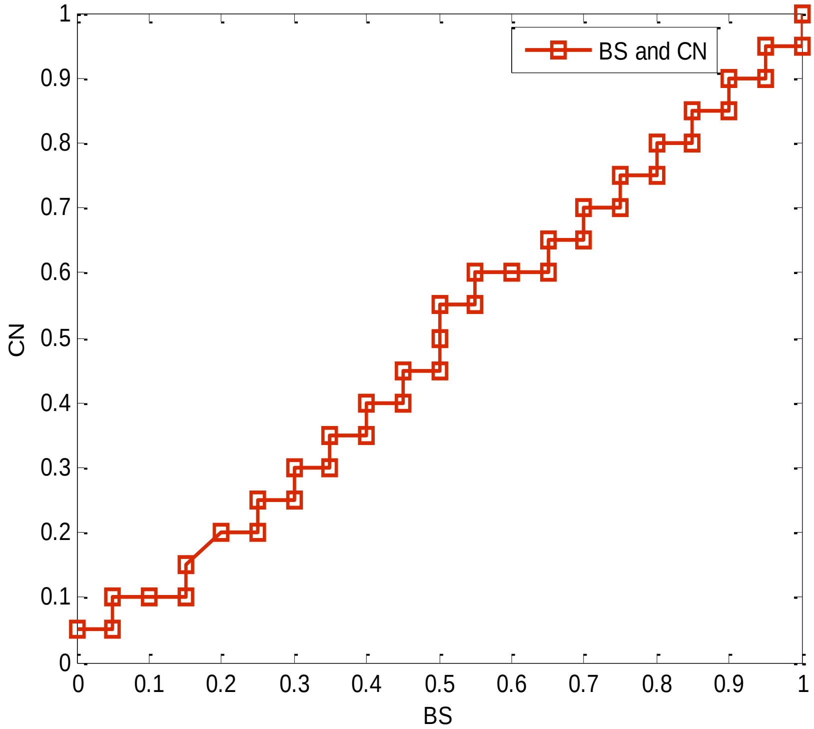

Figure 3: Graphical Representation of Black-Scholes and Crank-Nicolson using Kolmogorov-Smirnov.

The plot above shows that the two independent samples (BS and CN) are drawn from a common distribution; its normality assumptions are significant.

CONCLUSIONS

This paper considered Black-Scholes partial differential equation, its variants and interpretation of results following the methods in Section 2. From the results of BS and CN numerical solutions, it was noticed that BS and CN are identical and goodness of fit test was used to identify a class of probability distributions, the random process which generated the data.

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