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DYNAMIC RESPONSE OF UNIFORM CANTILEVER BEAMS ON ELASTIC FOUNDATION

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ABSTRACT: *The problem of the flexural vibrations of a uniform cantilever Bernoulli-Euler beams resting on an elastic foundation is studied in this paper. The analytical solution is based on the expression of the Heaviside function as a Fourier series and the fourth order partial differential equation of beam vibration under fixed and free end boundary conditions is transformed to second order ordinary differential equation by the generalized finite integral transform. The method of Struble's asymptotic technique is then used to simplify the resulting equation and make it amenable to the methods of Laplace and convolution theory. The effects of velocity of the load, axial force and flexural stiffness on the natural frequencies of the beam model are studied. The solutions obtained are verify first and then used to investigate the significance of different parameters on the beam behaviour.*

KEYWORDS: Material Damping Intensity, Flexural Stiffness, Critical Velocity, Resonance, Modified Natural Frequency.

INTRODUCTION

In recent years, the response of structural and flexible members to moving loads has received a lot of attention in the literature owing to its relevance in many diverse areas [1-6]. In particular, the response of elastic beam which is one of the structural engineering problems of theoretical and practical interest has provoked series of investigations. Due to the significance of this structural member, many researchers have dealt extensively with the study of dynamic characteristics of beams under the action of moving loads. Among the earlier researchers on the dynamic analysis of elastic beam was Ayre et al [7] who studied the effect of the ratio of the weight of the load to the weight of a simply supported beam for a constantly moving mass load. They obtained the exact solution for the resulting partial differential equation by using the infinite series method. Bolotin [8] used Galerkin's method in carrying out the dynamic analysis of the problem involving a concentrated mass traversing a simply supported beam moving at a constant speed. In a more recent development, Foda and Abduljabbar [9] worked on the dynamic green formulation for the response of a beam structure to a moving mass while Park et al [10] studied the natural frequencies and open-loop responses of an elastic beam fixed on a moving cart and carrying an intermediate mass. In same vein, Lin and Trethewey [11] solved the dynamic analysis of an elastic Bernoulli-Euler beam subjected to dynamic loads, induced by arbitrary movements of spring-mass-damper system, using finite element method (FEM). Olsson [12] presented a basic understanding of the moving load problem and reference data for more general studies. Jaiswal and Iyenggar [13] studied the

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dynamic response of an infinitely long beam resting on a foundation of finite depth and under the action of a moving force. The effects of various parameters such as foundation mass, velocity of the moving load, damping and axial force on the beam were investigated. Lee [14] used the Bernoulli-Euler beam theory and the assumed mode method to analyse the transverse vibration of a beam with intermediate point constraints subjected to a moving load. Fryba [15] presented the analytical solutions for simple problems of simply supported and continuous beams with uniform cross-section. A semi-analytic analysis based on the two-dimensional linear theory of elasticity for the transient dynamic response of a simply supported arbitrary elastic beam under the action of a transverse arbitrary distributed moving load is presented by Hasheminejad and Rafsanjani [16]. The dynamic response of non-uniform Bernoulli-Euler simply supported beam subjected to moving loads and rested on a nonlinear viscoelastic foundation was taken up by Abdelghany et al [17]. They investigated the system parameters and magnitude of the moving load effects on the vertical deflections of the non-uniform beam. In these investigations, however, only numerical or semi-analytical techniques have been employed to solve the governing equation due to the rigour of the load-structure interactions and complex nature of the resulting equations. Nevertheless, analytical solution is desirable as it sheds light on some vital information in the vibrating system. This paper therefore investigates the flexural vibrations of a cantilever beam under uniformly distributed masses. Both gravity and inertia effects of the uniformly distributed masses are taken into consideration and the beam is taken to rest of Winkler foundation. The solution technique which is analytical involves using the generalized finite integral transform, the expression of the Heaviside function as a Fourier series and the use of the modified Struble's asymptotic technique to solve the problem of the flexural vibrations of a Bernoulli-Euler beam under fixed and free end boundary conditions. In addition, conditions under which resonance is reached, is also obtained. The beam mass intensity, material damping intensity and flexural stiffness are assumed constant along the beam length.

MATHEMATICAL MODEL

Consider a uniform cantilever beam with a constant flexural rigidity EI, length L resting on an elastic foundation of stiffness K. The mass M is assumed to touch the beam at time $t=0$ and travel across with a constant velocity v.

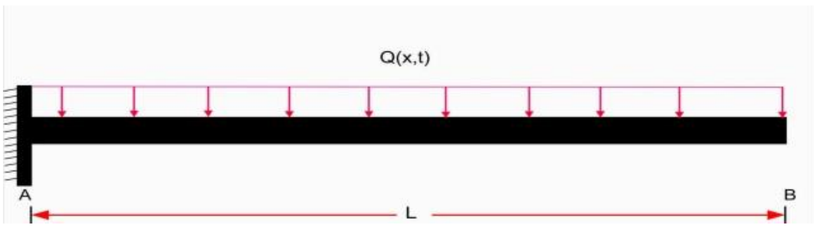

Figure 1. Cantilever beam subjected to distributed load.

The equation of motion of the damped beam is given by the fourth order partial differential equation

$$EIZ^{iv}(x,t) - NZ''(x,t) + \overline{m}\ddot{Z}(x,t) + C\dot{Z}(x,t) + KZ(x,t) = Q(x,t) \tag{1}$$

Dots and primes denote partial derivatives with respect to time t and position coordinate x, respectively.

The corresponding boundary conditions are

$$Z(0,t)=0 \qquad Z'\psi(0,t)=0 \tag{2}$$

$$Z''(L,t)=0 \qquad Z'''(L,t)=0 \tag{3}$$

Considering the inertia effect of the moving load, the load $Q(x,t)$ takes the form

$$Q(x,t)=MgH[x-vt]\left[1-\frac{1}{g}\rho Z(x,t)\right] \tag{4}$$

g is the acceleration due to gravity and ρ is the convective acceleration operator defined as

$$\rho=\frac{\partial^2}{\partial t^2}+2v\frac{\partial^2}{\partial x\partial t}+v^2\frac{\partial^2}{\partial x^2} \tag{5}$$

and the time t is assumed to be limited to that interval of time within the mass of the beam, that is

$$0\le vt\le L \tag{6}$$

Substituting Eqs. (4) and (5) into Eq. (1) gives

$$\begin{aligned}&EIZ^{iv}(x,t)-NZ''(x,t)+\overline{m}\ddot{Z}(x,t)+C\dot{Z}(x,t)+KZ(x,t)\\&+MH[x-vt]\left[\ddot{Z}(x,t)+2v\dot{Z}'(x,t)+v^2Z''(x,t)\right]=MgH[x-vt]\end{aligned} \tag{7}$$

In an attempt to solve Eq. (7) an approximate series solution is sought. Here, we employ the generalized finite integral transform defined as follows:

$$Z(m,t)=\int_0^L Z(x,t)\psi_m(x)dx \tag{8}$$

where

$$Z(x,t)=\sum_{m=1}^{\infty}\frac{\mu}{\sigma_m}Z(m,t)\psi_m(x) \tag{9}$$

$\psi_m(x)$ is the general kernel chosen so that the clamped-free end boundary conditions are satisfied; and ψ_m is defined as:

$$\sigma_m=\int_0^L \mu\psi_m^2(x)dx \tag{10}$$

Thus, the normal mode of vibration of the beam is as

$$\psi_m(x) = Sin\frac{\theta_m x}{L} + A_m Cos\frac{\theta_m x}{L} + B_m Sinh\frac{\theta_m x}{L} + C_m Cosh\frac{\theta_m x}{L} \tag{11}$$

where A_m, B_m, C_m are constants that can be determined using the boundary condition and θ_m is the mode frequency.

By applying the generalized finite integral transform (8), Eq. (7) becomes

$$\begin{aligned} &EIB(0,L,t) + EIH_A(t) - NH_B(t) + \overline{m}\ddot{Z}(m,t) + C\dot{Z}(m,t) \\ &+ KZ(m,t) + H_C(t) + H_D(t) + H_E(t) = \frac{Mg}{\overline{m}} H_F(t) \end{aligned} \tag{12}$$

where

$$H_A(t) = \int_0^L Z(x,t)\psi_m^{iv}(x)dx; \qquad H_B(t) = \int_0^L Z''(x,t)\psi_m(x)dx$$

$$H_C(t) = M\int_0^L H[x-vt]\ddot{Z}(x,t)\psi_m(x)dx; \qquad H_D(t) = 2Mv\int_0^L H[x-vt]\dot{Z}'(x,t)\psi_m(x)dx$$

$$H_E(t) = Mv^2\int_0^L H[x-vt]Z''(x,t)\psi_m(x)dx; \qquad H_F(t) = \int_0^L H[x-vt])\psi_m(x)dx \tag{13}$$

The ordinary differential Eq. (12) is valid for cantilever end conditions. Using the Fourier series representation of the Heaviside unit step function, namely

$$H[x-vt] = \frac{1}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2r+1)\pi(x-vt)}{2r+1}, \qquad 0 < x < L \tag{14}$$

simplifying integrals (13) in conjunction with Eq. (14), Eq. (12) after some simplifications yield

$$\begin{aligned} &\ddot{Z}(m,t) + \frac{C}{\overline{m}}\dot{Z}(m,t) + \left(\omega_m^2 + \frac{K}{\overline{m}}\right)Z(m,t) - \frac{N}{\overline{m}}\sum_{k=1}^{\infty} R_A(k,m)Z(k,t) + \Gamma_0\left\{\left(\frac{1}{4}R_B(k,m)\right.\right. \\ &\left.+ \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_C(r,k,m) - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2r+1)\pi vt}{2r+1}R_D(r,k,m)\right)\ddot{Z}(k,t) \\ &+\left(\frac{v}{2}R_E(k,m) + \frac{2v}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_F(r,k,m) - \frac{2v}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_G(r,k,m)\right)\dot{Z}(k,t) \\ &\left.\left.+\left(\frac{v^2}{4}R_A(k,m) + \frac{v^2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_H(r,k,m) - \frac{v^2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_I(r,k,m)\right)Z(k,t)\right\}\right\} \\ &= \frac{MgL}{\overline{m}\lambda_m}\left[-\cos\lambda_m + A_m\sin\lambda_m + B_m\cosh\lambda_m + C_m\sinh\lambda_m + \cosh\frac{\lambda_m vt}{L} - A_m\sin\frac{\lambda_m vt}{L}\right. \\ &\left. - B_m\cosh\frac{\lambda_m vt}{L} - C_m\sinh\frac{\lambda_m vt}{L}\right] \end{aligned} \tag{15}$$

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$$\omega_m^2 = \frac{\lambda_m^4}{L^4}\frac{EI}{\overline{m}} \qquad \Gamma_0 = \frac{M}{\overline{m}L}, \tag{16}$$

$$R_A(k,m) = \frac{1}{\tau_k}\int_0^L \psi_k''(x)\psi_m dx; \qquad R_B(k,m) = \frac{1}{\tau_k}\int_0^L \psi_k(x)\psi_m(x)dx$$

$$R_C(r,k,m) = \frac{1}{\tau_k}\int_0^L \sin(2r+1)\pi x\,\psi_k(x)\psi_m(x)dx; \quad R_D(r,k,m) = \frac{1}{\tau_k}\int_0^L \cos(2r+1)\pi x\,\psi_k(x)\psi_m(x)dx$$

$$R_E(k,m) = \frac{1}{\tau_k}\int_0^L \psi_k'(x)\psi_m(x)dx; \qquad R_F(r,k,m) = \frac{1}{\tau_k}\int_0^L \sin(2r+1)\pi x\,\psi_k'(x)\psi_m(x)dx$$

$$R_G(r,k,m) = \frac{1}{\tau_k}\int_0^L \cos(2r+1)\pi x\,\psi_k'(x)\psi_m(x)dx; \quad R_H(r,k,m) = \frac{1}{\tau_k}\int_0^L \sin(2r+1)\pi x\,\psi_k''(x)\psi_m(x)dx$$

$$R_I(r,k,m) = \frac{1}{\tau_k}\int_0^L \cos(2r+1)\pi x\,\psi_k''(x)\psi_m(x)dx \tag{17}$$

Eq. (13) is the fundamental equation of our problem when the beam is resting on Winkler foundation has a cantilever end supports.

Closed Form Solution

Case I: The differential equation describing the flexural vibrations of the finite beam subjected to a moving distributed force and may be obtained from Eq. (15) by setting $\Gamma_0 = 0$

$$\begin{aligned}
&\ddot{Z}(m,t) + \frac{C}{\overline{m}}\dot{Z}(m,t) + \left(\omega_m^2 + \frac{K}{\overline{m}}\right)Z(m,t) - \frac{N}{\overline{m}L}\sum_{k=1}^{\infty} LR_A(k,m)Z(k,t) \\
&= \frac{MgL}{\overline{m}\lambda_m}\Big[-\cos\lambda_m + A_m\sin\lambda_m + B_m\cosh\lambda_m + C_m\sinh\lambda_m + \cosh\frac{\lambda_m vt}{L} - A_m\sin\frac{\lambda_m vt}{L} \\
&- B_m\cosh\frac{\lambda_m vt}{L} - C_m\sinh\frac{\lambda_m vt}{L}\Big]
\end{aligned} \tag{18}$$

Eq. (18) is rearranged to take the form

$$\begin{aligned}
&\ddot{Z}(m,t) + \omega_{mf}^2 Z(m,t) + F_1\dot{Z}(m,t) - \varepsilon^* F_2\sum_{\substack{k=1\\k\neq m}}^{\infty} LR_A(k,m)Z(k,t) \\
&= \frac{MgL}{\overline{m}\lambda_m}\left[\Psi_m + \cos\alpha vt - A_m\sin\alpha vt - B_m\cosh\alpha vt - C_m\sinh\alpha vt\right]
\end{aligned} \tag{19}$$

where

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$$\omega_{mf}^2 = \omega_m^2 + \frac{K}{\bar{m}} + \varepsilon_1^* F_2 Q_B(m,m), \quad \varepsilon_1^* = \frac{N}{L}, \quad F_1 = \frac{C}{\bar{m}}, \quad F_2 = \frac{L}{\bar{m}}, \quad \alpha = \frac{\lambda_m}{L} \tag{20}$$

$$R_A(m,m) = R_A(k,m)\big|_{k=m}, \qquad \Psi_m = \cos\lambda_m + A_m \sin\lambda_m + B_m \cosh\lambda_m + C_m \sinh\lambda_m \tag{21}$$

The modified frequency corresponding to the frequency of the free system due to the presence of the effect of axial force is sought. Thus, we set the right hand side of (19) to zero and considered a parameter $\eta << 1$ for any arbitrary ratio ε^*, defined as

$$\Omega = \frac{\varepsilon_1^*}{1+\varepsilon_1^*} \tag{22}$$

so that

$$\varepsilon_1^* = \Omega + O(\Omega^2) \tag{23}$$

Substituting Eq. (23) in the homogeneous part of Eq. (19) one obtains

$$\begin{aligned}&\ddot{Z}(m,t) + \omega_{mf}^2 Z(m,t) + F_1 \dot{Z}(m,t) - \Omega F_2 \sum_{\substack{k=1 \\ k\neq m}}^{\infty} L R_A(k,m) Z(k,t) \\ &= \frac{MgL}{\bar{m}\lambda_m}\left[\Psi_m + \cos\alpha v t - A_m \sin\alpha v t - B_m \cosh\alpha v t - C_m \sinh\alpha v t\right]\end{aligned} \tag{24}$$

The axial force effect is regarded as negligible and η is set to zero in Eq. (24), then the solution of (24) becomes

$$Z(m,t) = \vartheta(m,t)\cos[\omega_{mf} t - \beta(m,t)] \tag{25}$$

where $\vartheta(m,t)$, ω_{mf}, $\beta(m,t)$ are constants.

As $\Omega << 1$, Struble's technique requires that the asymptotic solutions of the homogeneous part of equation (24) be of the form

$$Z(m,t) = \vartheta(m,t)\cos\left[\omega_{mf} t - \beta(m,t)\right] + \eta Z_1 + O(\Omega^2) \tag{26}$$

where $\vartheta(m,t)$ and $\beta(m,t)$ are slowly varying functions of time.

Eq. (26) and its derivatives are substituted into Eq. (24) and neglecting terms which do not contribute to the variational equations, one obtains

$$\begin{aligned}&-2\dot{\vartheta}(m,t)\omega_{mf} \sin\left[\omega_{mf} t - \beta(m,t)\right] + 2\vartheta(m,t)\omega_{mf}\dot{\beta}(m,t)\cos\left[\omega_{mf} t - \beta(m,t)\right] \\ &+ F_1 \dot{\vartheta}(m,t)\cos\left[\omega_{mf} t - \beta(m,t)\right] - \vartheta(m,t)\omega_{mf} \sin\left[\omega_{mf} t - \beta(m,t)\right] \\ &+ \vartheta(m,t)\omega_{mf}^2 \cos\left[\omega_{mf} t - \beta(m,t)\right] = 0\end{aligned} \tag{27}$$

retaining terms to $O(\Omega)$ only.

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The variational equations are obtained Eq. (27) as

$$-2\dot{\vartheta}(m,t)\omega_{mf} - \vartheta(m,t)\omega_{mf} = 0 \tag{28}$$

and

$$2\vartheta(m,t)\omega_{mf}\dot{\beta}(m,t) + F_1\dot{\vartheta}(m,t) + \vartheta(m,t)\omega_{mf}^2 = 0 \tag{29}$$

From Eqs. (28) and (29)

$$\vartheta(m,t) = Ae^{-\frac{1}{2}t} \tag{30}$$

and

$$\beta(m,t) = \frac{1}{2\omega_{mf}}\left(\frac{1}{2}F_1 - \omega_{mf}\right) + \chi_m \tag{31}$$

The first approximation to the homogeneous system is

$$Z(m,t) = Ae^{-\frac{1}{2}t}\cos\left[\aleph_{aj}t - \chi_m\right] \tag{32}$$

where

$$\aleph_{aj} = \omega_{mf}\left[1 - \frac{1}{2}\left(\frac{1}{2\omega_{mf}^2}F_1 - \frac{1}{\omega_{mf}}\right)\right] \tag{33}$$

is the modified natural frequency corresponding to the frequency of the free system due to the presence of the effect of axial force.. It is observed that when $\Omega = 0$, we recover the frequency of the moving force problem when the axial force effect of the beam is neglected.

In order to solve the non-homogeneous Eq. (18), the differential operator which acts on $Z(m,t)$ is replaced by the equivalent free system operator defined by the modified frequency $\aleph_{aj}$. Thus

$$\ddot{Z}(m,t) + \aleph_{aj}^2 Z(m,t) = 0 \tag{34}$$

Therefore, the moving force problem is reduced to the non-homogeneous ordinary differential equation given by

$$\ddot{Z}(m,t) + \aleph_{aj}^2 Z(m,t) = \frac{MgL}{\overline{m}m\pi}\left[\Psi_m + \cos\alpha vt - A_m \sin\alpha vt - B_m \cosh\alpha vt - C_m \sinh\alpha vt\right] \tag{35}$$

When Eq. (35) is solved in conjunction with the initial conditions, one obtains expression for $Z(m,t)$. Thus in view of Eq. (9)

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$$Z(x,t)=\frac{1}{\tau_m}\sum_{m=1}^{\infty}\frac{\mu MgL}{(\alpha v^4-\aleph_{aj}^4)\overline{m}m\pi}\Bigg\{\Psi_m\frac{(\alpha v^4-\aleph_{aj}^4)(1-\cos\aleph_{aj}t)}{\aleph_{aj}}+(\alpha v^2+\aleph_{aj}^2)(\cos\alpha vt-\cos\aleph_{aj}t)$$
$$-\left[\aleph_{aj}^2\cos 2\aleph_{aj}t\cosh\alpha vt-\alpha v^2\cosh\alpha vt+\aleph_{aj}\alpha v\sin 2\aleph_{aj}t\sinh\alpha vt+\frac{\left(\alpha v^2-\aleph_{aj}^2\right)\cos\aleph_{aj}t}{\aleph_{aj}}\right]$$
$$+A_m\frac{(\alpha v^2+\aleph_{aj}^2)}{\aleph_{aj}}\left[\aleph_{aj}\sin\alpha vt-\alpha v\sin\aleph_{aj}t\right]-C_m\Big[\cosh\alpha vt-\cos 2\aleph_{aj}t+\alpha v^2\sinh\alpha vt$$
$$+\aleph_{aj}\alpha v\sin 2\aleph_{aj}t\sinh\alpha vt-\frac{\alpha v\left(\alpha v^2-\aleph_{aj}^2\right)\sin\aleph_{aj}t}{\aleph_{aj}}\Big]\Bigg\}\Big[\sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}$$
$$+B_m\sinh\frac{\lambda_m x}{L}+C_m\cosh\frac{\lambda_m x}{L}\Big] \tag{36}$$

The equation (36) represents the transverse displacement response to a moving force of a prestressed uniform Bernoulli-Euler beam resting on constant Winkler elastic foundation and having a cantilever end supports.

Case II: If the mass of the moving load is commensurable with that of the structure, the inertia effect of the moving load is not negligible. Thus $\Gamma_0 \neq 0$ and one is required to solve the entire Eq. (13) when no term of the coupled differential equation is neglected. This is termed moving mass problem.

Thus, Eq. (13) can be rewritten in the form

$$\ddot{Z}(m,t)+\frac{\Gamma_0F_1\left(\frac{v}{2}R_E(k,m)+\frac{2v}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_F(r,k,m)-\frac{2v}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_G(r,k,m)\right)}{1+\Gamma_0\left(\frac{1}{4}R_B(k,m)+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_C(r,k,m)-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2r+1)\pi vt}{2r+1}R_D(r,k,m)\right)}\dot{Z}(m,t)$$
$$-\frac{\aleph_{aj}^2+\Gamma_0\left(\frac{v^2}{4}R_A(k,m)+\frac{v^2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_H(r,k,m)-\frac{v^2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_I(r,k,m)\right)}{1+\Gamma_0\left(\frac{1}{4}R_B(k,m)+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_C(r,k,m)-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2r+1)\pi vt}{2r+1}R_D(r,k,m)\right)}Z(m,t)$$
$$=\frac{MgL\mathrm{T}_{MM}}{\overline{m}m\pi\left\{1+\Gamma_0\left(\frac{1}{4}R_B(k,m)+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_C(r,k,m)-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2r+1)\pi vt}{2r+1}R_D(r,k,m)\right)\right\}} \tag{37}$$

where

$$\mathrm{T}_{MM} = -\cos\lambda_m + A_m \sin\lambda_m + B_m \cosh\lambda_m + C_m \sinh\lambda_m + \cosh\frac{\lambda_m vt}{L} - A_m \sin\frac{\lambda_m vt}{L} - B_m \cosh\frac{\lambda_m vt}{L} - C_m \sinh\frac{\lambda_m vt}{L} \tag{38}$$

Considering the homogeneous part of Eq. (37) and going through the same arguments and analysis as in previous case, the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass is

$$\Im_{MM} = \aleph_{aj}\left[1-\frac{1}{2}\left(F_1 \mathrm{H}_A(r,k,m) - \frac{\Gamma_1 \mathrm{H}_B(r,k,m)}{\aleph_{aj}^2}\right)\right] \tag{39}$$

where

$$\mathrm{H}_A(r,k,m) = \frac{v}{2}R_E(k,m) + \frac{2v}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_F(r,k,m) - \frac{2v}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_G(r,k,m) \tag{40}$$

$$\mathrm{H}_B(r,k,m) = \frac{v^2}{4}R_A(k,m) + \frac{v^2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_H(r,k,m) - \frac{v^2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2r+1)\pi vt}{2r+1}R_I(r,k,m) \tag{41}$$

Thus, to solve the non-homogeneous Eq. (37), the differential operator which acts on $Z(m,t)$ is replaced by the equivalent free system operator defined by the modified frequency $\Im_{MM}$. That is

$$\ddot{Z}(m,t) + \Im_{MM}^2 Z(m,t) = G_0\left[-\cos\lambda_m + A_m \sin\lambda_m + B_m \cosh\lambda_m + C_m \sinh\lambda_m + \cosh\frac{\lambda_m vt}{L} - A_m \sin\frac{\lambda_m vt}{L} - B_m \cosh\frac{\lambda_m vt}{L} - C_m \sinh\frac{\lambda_m vt}{L}\right] \tag{42}$$

It is of note that Eq. (42) is analogous to Eq. (35) with $\Im_{MM}$ replacing $\aleph_{aj}$. Therefore, when Eq. (42) is solved in conjunction with the initial conditions, one obtains expression for $Z(m,t)$ and in view of Eq. (9), one obtains

$$Z(x,t)=\frac{1}{\tau_m}\sum_{m=1}^{\infty}\frac{\mu G_0}{(\alpha v^4-\Im_{MM}^4)}\left\{\Psi_m\frac{(\alpha v^4-\Im_{MM}^4)(1-\cos\Im_{MM}t)}{\Im_{MM}}+(\alpha v^2+\Im_{MM}^2)(\cos\alpha vt-\cos\Im_{MM}t)\right.$$
$$-\left[\Im_{MM}^2\cos 2\Im_{MM}t\cosh\alpha vt-\alpha v^2\cosh\alpha vt+\Im_{MM}\alpha v\sin 2\Im_{MM}t\sinh\alpha vt+\frac{(\alpha v^2-\Im_{MM}^2)\cos\Im_{MM}t}{\Im_{MM}}\right]$$
$$+A_m\frac{(\alpha v^2+\Im_{MM}^2)}{\Im_{MM}}\left[\Im_{MM}\sin\alpha vt-\alpha v\sin\Im_{MM}t\right]-C_m\left[\cosh\alpha vt-\cos 2\Im_{MM}t+\alpha v^2\sinh\alpha vt\right.$$
$$\left.\left.+\Im_{MM}\alpha v\sin 2\Im_{MM}t\sinh\alpha vt-\frac{\alpha v(\alpha v^2-\Im_{MM}^2)\sin\Im_{MM}t}{\Im_{MM}}\right]\right\}\left[\sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}\right.$$
$$\left.+B_m\sinh\frac{\lambda_m x}{L}+C_m\cosh\frac{\lambda_m x}{L}\right] \tag{43}$$

Eq. (43) is the transverse displacement response to a moving mass of a Bernoulli-Euler beam resting on constant Winkler elastic foundation and having a cantilever end conditions.

Discussion of the Closed Form Solutions

When an undamped system such as this is studied, one is interested in the resonance conditions of the vibrating system, because the transverse displacement of the elastic beam may increase without bound. Thus, for the illustrate example, we observe that the Bernoulli-Euler beam traversed by distributed moving force reaches a state of resonance whenever

$$\aleph_{aj}=\alpha v \tag{44}$$

while the same beam under the action of a distributed moving mass experiences resonance effect whenever

$$\Im_{MM}=\alpha v \tag{45}$$

Evidently,

$$\Im_{MM}=\aleph_{aj}\left\{1-\frac{1}{2}\left[F_1\mathrm{H}_A(r,k,m)-\frac{\Gamma_1\mathrm{H}_B(r,k,m)}{\aleph_{aj}^2}\right]\right\} \tag{46}$$

Eqs. (44) and (46) show that for the same natural frequency, the critical speed for the same system consisting of a uniform Bernoulli-Euler beam resting on an elastic foundation and traversed by a distributed moving force is greater than that traversed by a distributed moving mass. Thus resonance is reached earlier in the distributed moving mass system than in the distributed moving force system.

Illustrative Example

In this section, practical example of classical boundary conditions of cantilever end conditions is presented in this paper.

Cantilever end conditions

In this case the beam type structure is clamped at one end and free at the other end. Accordingly, the boundary conditions are

$$Z(0,t)=0=\dot{Z}(0,t), \qquad \ddot{Z}(L,t)=0=\dddot{Z}(L,t) \tag{47}$$

and for normal modes

$$U_m(0)=0=\dot{U}_m(0), \qquad \ddot{U}_m(L)=0=\dddot{U}_m(L) \tag{48}$$

which implies that

$$U_k(0)=0=\dot{U}_k(0), \qquad \ddot{U}_k(L)=0=\dddot{U}_k(L) \tag{49}$$

It can be shown that

$$A_m = \frac{\sin\lambda_m - \sinh\lambda_m}{\cos\lambda_m - \cosh\lambda_m} = \frac{\cos\lambda_m - \cosh\lambda_m}{\sinh\lambda_m + \sin\lambda_m} = -C_m \text{ and } B_m = -1 \tag{50}$$

at end $x=0$ and at $x=L$

$$A_m = \frac{-\sin\lambda_m - \sinh\lambda_m}{\cos\lambda_m + \cosh\lambda_m} = \frac{-\cos\lambda_m - \cosh\lambda_m}{\sinh\lambda_m - \sin\lambda_m} = -C_m \text{ and } B_m = -1 \tag{51}$$

and the frequency equation for both end conditions is

$$\cos\lambda_m \cosh\lambda_m = -1 \tag{52}$$

and we have that

$$\lambda_1 = 1.875, \quad \lambda_2 = 4.694, \quad \lambda_3 = 7.855 \tag{53}$$

using (50), (51) and (53) in equations (36) and (43), one obtains the displacement response respectively to a moving force and moving mass of a cantilever Rayleigh beam resting on a Winkler elastic foundation.

NUMERICAL RESULTS AND DISCUSSION

Consider a cantilever beam resting on an elastic foundation subjected to a moving mass with the following properties of the beam adopted as $L=5m$, $\overline{m}=2758.291kg/m^3$, $E=3.1\times10^{10}N/m^2$, $I=2.87698\times10^{-3}m^4$ and $v=20mls$. The values of axial force N is varied

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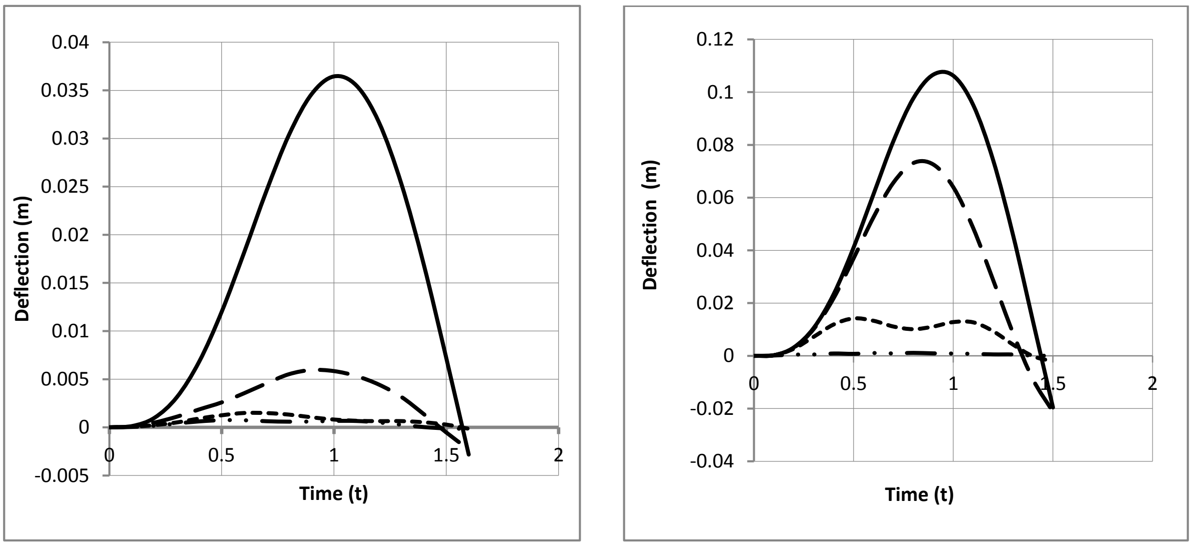
between $0N$ and $200000N$, material damping intensity is varied from 0 to 0.75 and foundation stiffness is varied between $0N/m^3$ and $4000000\,N/m^3$.

Figure 2 displays the displacement response of a uniform cantilever beam to both moving distributed force and moving distributed mass for various values of axial force N and fixed values of material damping intensity 0.5 and foundation stiffness $K = 40000$. The figures show that as N increases the response amplitude of the cantilever beam decrease. The effects of material damping intensity on the vibration amplitude are indicated in figure 3. As it is expected, the beam is affected significantly by increase in various values of C which increases the amplitudes of the beam for both moving distributed force and moving distributed mass. Figure 4 shows the effect of beam-foundation system on the response amplitude of both moving distributed force and moving distributed mass for different foundation stiffness and fixed values of $N = 20000$ and $C = 0.5$. It is shown that as K increases the response amplitude decreases because the system becomes more rigid. Hence the existence of foundation stiffness increases the overall rigidity of the beam-foundation system. In figure 5(a), the relationship between the critical velocity and the axial force N is displayed. It is shown from the figure that as N increases the critical velocity of the system also increases. Figure 5(b) clearly shows that increase in the mass ratio increases the critical velocity of the system while figure 6(a) depicts that as the velocity of the traversing load increases the modified natural frequency of the system also increases. Finally, in figure 6(b) the comparison of the displacement response of the moving distributed force and moving distributed mass cases of the cantilever beam for fixed values of axial force $N = 200000$, foundation stiffness $K = 400000$ and material damping intensity $C = 0.75$ is displayed. Clearly, the response amplitude of the moving distributed mass system is higher than that of moving distributed force system.



(a) (b)

Figure 2: Dynamic deflections of cantilever beam for different axial force N:
(a) distributed moving force. (b) distributed moving mass
(---------- $N = 0$; -- -- -- $N = 20000$; - - - - - $N = 100000$; —· ·—· · $N = 200000$)

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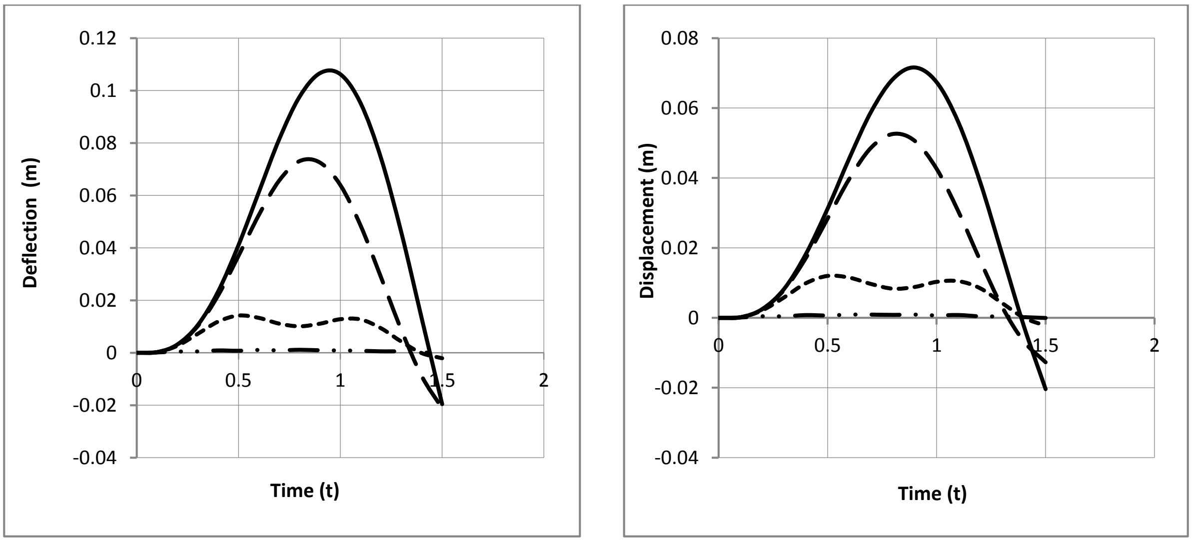
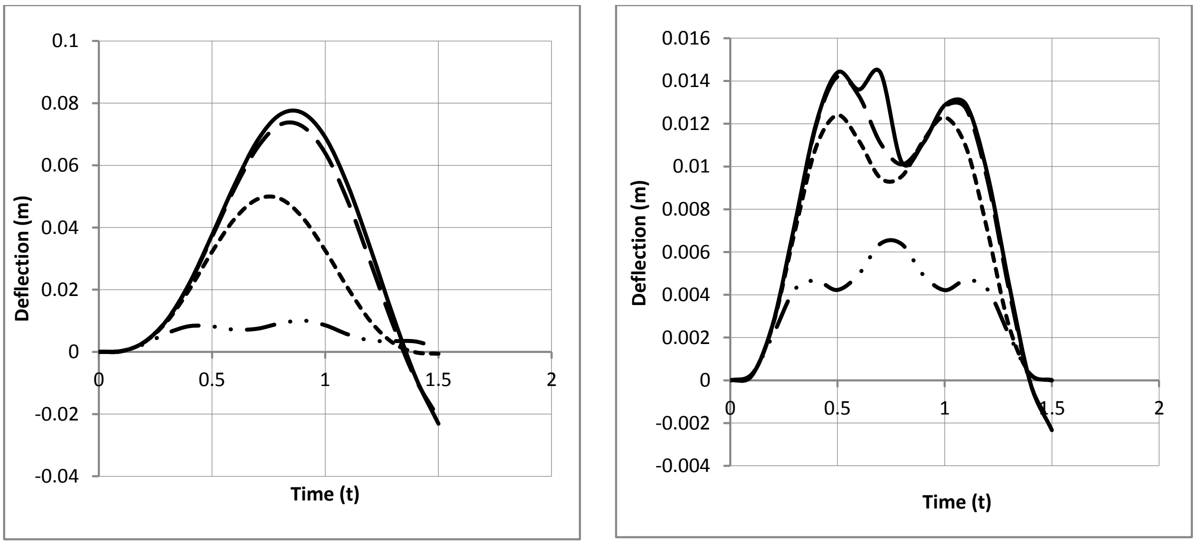


(a) (b)

Figure 3: Dynamic deflections of cantilever beam for different material damping intensity C :

(a) distributed moving force. (b) distributed moving mass

(---------- $C=0$; -- -- -- $C=0.2$; - - - - - $C=0.5$; —· ·—· · $C=0.75$)

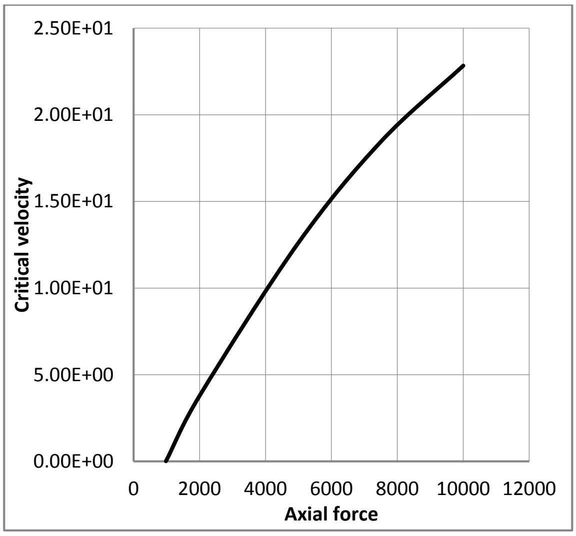


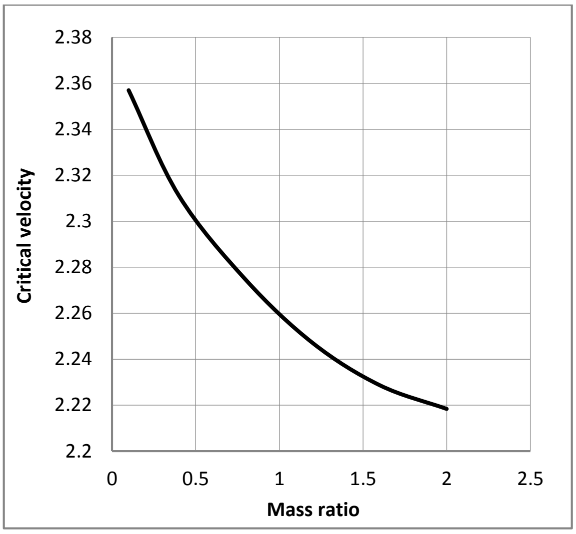
(a) (b)

Figure 4: Dynamic deflections of cantilever beam for different flexural stiffness K :

(a) distributed moving force. (b) distributed moving mass

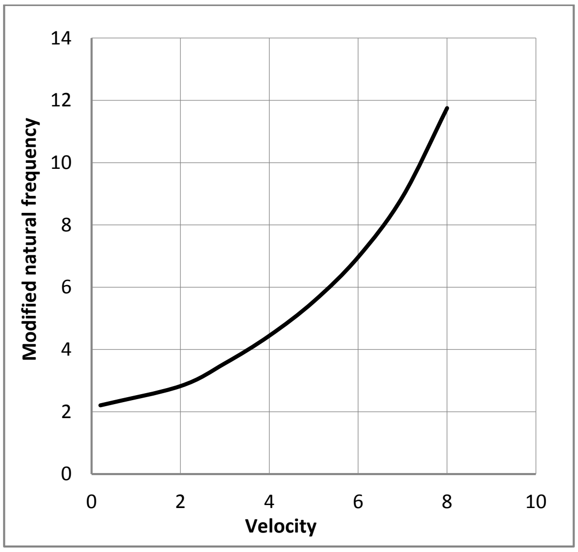
(---------- $K=0$; -- -- -- $K=40000$; - - - - - $K=400000$; —· ·—· · $K=4000000$)

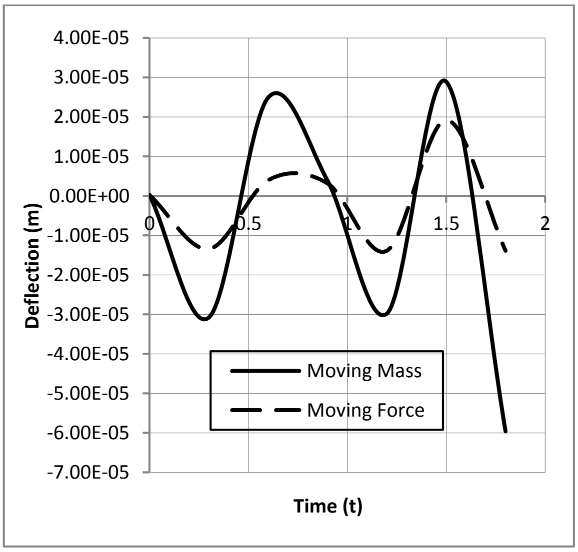




(a) (b)

Figure 5: Variation of critical velocity versus: (a) axial force. (b) mass ratio





(a) (b)

Figure 6: Comparison between the: (a) distributed moving force (b) distributed moving mass

CONCLUSION

In this paper, the analytical expressions for the vibrational behaviour of uniform flexural cantilever beam resting on a Winkler type elastic foundation has been investigated. The dynamic equation of the beam-foundation systems is solved by the method of solution based on generalized finite integral transform and the Struble's asymptotic technique. An approximate analytical solution of the governing fourth order partial differential equation of the beam-foundation system is presented. The numerical analyses carried out show that as the flexural stiffness K increases, the response amplitudes of the cantilever beam decreases. In addition, the influence of variation in the axial force N and damping intensity C on the response of the beam were studied. Analytical solutions further show that for the same natural frequency, the critical speed for the system traversed by uniformly distributed moving forces at constant speed is greater than that of the uniformly distributed moving mass problem. Hence, resonance is reached earlier in the moving distributed mass system. Finally, this work has suggested valuable method of approximate analytical solution for this class of problem for beam supported at one end with the other end over-hanged and free.

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