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ON THE GLOBAL EXISTENCE OF SOLUTION OF THE COMPARISON SYSTEM AND VECTOR LYAPUNOV ASYMPTOTIC EVENTUAL STABILITY FOR NONLINEAR IMPULSIVE DIFFERENTIAL SYSTEMS

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ABSTRACT: *In this paper, the existence of maximal solution of the comparison system as well as the asymptotic eventual stability of nonlinear impulsive differential equations with fixed moments of impulse is examined using the vector Lyapunov functions which is generalized by a class of piecewise continuous Lyapunov functions. It was distinctly established that the maximal solution of the comparison system majorizes the vector form of the Lyapunov functions. The novelty in the use of the vector Lypunov functions lies in the fact that the "restrictions" encountered by the scalar Lyapunov function is safely handled especially for large scale dynamical systems, since it involves splitting the Lyapunov functions into components so that each of the components can easily describe the behavior of the solution state. Together with comparison results, sufficient conditions for eventual stability and asymptotic are presented.*

KEYWORDS AND PHRASES: Asymptotic eventual stability; impulsive differential equation; vector Lyapunov functions.

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INTRODUCTION

In the qualitative theory of differential equations, one of the main properties of interest is the stability of solutions, as it enables us to compare the behavior of solutions starting at different points [1].

Now, the stability of solutions of differential equations via Lyapunov method has been intensively investigated in the past, and in many real cases, it is obligatory to study the stability of such sets, which are invariant with respect to a given system of differential equation. This immediately excludes the stability in the sense of Lyapunov [23]. To allay the problem that will arise subsequently, [14] introduced a new concept called eventual stability, maintaining that, the set under consideration, despite not being invariant in the usual sense, is invariant in the asymptotic sense (See also [25]).

The theory of impulsive differential equation (IDE) is richer than the corresponding theory of differential equations [11], as they constitute very important models for describing the true state of several real life processes and phenomena.

Of course, many evolution processes are characterized by the fact that at certain moments of time they undergo a change of state abruptly. These processes are subject to short term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects [11].

Now, the efficient applications of impulsive differential system require the finding of criteria for stability of their solutions [21], and one of the most versatile methods in the study of the stability properties of impulsive systems is the method of Lyapunov functions.

There are several approaches in the literature in the study of the stability of solutions, one of which is the Lyapunov's second method. The novelty of the method over other methods of examining stability properties like the Razumikhin technique, the use of matrix inequality, variational method, Banach fixed point theory, monotone iteration method, etc. stems from the fact that the method allows us to examine the stability of solutions without first solving the given differential system. It involves seeking an appropriate continuously differentiable Lyapunov function that is positively definite and whose time derivative along the solution path is negative semidefinite. The stability of the trivial solution of impulsive differential equations has been extensively studied in [3, 6, 20, 27-35].

In this paper, the eventual stability of the system of nonlinear impulsive differential equation is examined. By employing the vector Lyapunov functions, generalized by a class of piecewise continuous Lyapunov functions together with the comparison results, sufficient conditions for the eventual stability of the solution is established with illustrative example.

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Preliminary notes and Definitions

Let R^n be the n-dimensional Euclidean space with norm $\|.\|$. Let Ω be a domain in R^n containing the origin; $R_+ = [0,\infty)$, $R = (-\infty,\infty)$, $t_0 \in R_+$, $t > 0$.

Let $J \subset R_+$. Define the following class of functions is a piecewise continuous function with points of discontinuity $t_k \in J$ at which $\alpha(t)$ exists.

Consider the impulsive differential system

$$\begin{aligned} x' &= f(t,x), t \neq t_k,\ t \geq t_0 \\ \Delta x &= I_k(x),\ t = t_k, k \in N \\ x(t_0^+) &= x_0 \end{aligned} \tag{2.1}$$

under the following assumptions:

A_0 (i) $0 < t_1 < t_2 < ... < t_k < ...,$ and $t_k \to \infty$ $k \to \infty$;

(ii) $f : R_+ \times R^n \to R^n$ is continuous in $(t_{k-1,}\ t_k] \times R^N$ and for each $x \in R^n$, $k = 1,2,...,$

$$\lim_{(t,y)\to(t_k{}^+,x)} f(t,y) = f(t_k{}^+, x) \text{ exists;}$$

(iii) $I_k : R^n \to R^n$

In this paper, we assume that the function f is Lipschitz continuous with respect to its second argument, and $f(t,0) \equiv 0$, $I_k(0) \equiv 0$ for all k, so that we have the trivial solution for (2.1), and the points $t_k, k = 1,2,...$ are fixed such that $0 < t_1 < t_2 < ...$ and $\lim_{k\to\infty} t_k = \infty$. The system (2.1) with initial condition $x(t_0) = x_0$ is assumed to have a solution $x(t; t_0, x_0) \in PC([t_0,\infty), R^N)$. Note that some sufficient conditions for the existence and uniqueness of the global solutions to (2.1) are considered in [9, 15, 16, 18, 26].

Remark 2.1.

The second equation in (2.1) is called the impulsive condition, and the function $I_k(x(t_k))$ gives the amount of jump of the solution at the point t_k.

Definition 2.1.

Let $V : R_+ \times R^N \to R_+^N$ be a continuous mapping of $R_+ \times R^N$ into R_+^N. Then V is said to belong to class L if,

(i) V is continuous in $(t_{k-1,}\ t_k] \times R^N$ and for each $x \in R^N$, $k = 1,2,...,$ and $\lim_{(t,y)\to(t_k{}^+,x)} V(t,y) = V(t_k{}^+, x)$ exists;

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(ii) V is locally Lipschitz with respect to its second argument x. For $(t_{k-1,} t_k] \times R^N$, we define the upper right Dini derivative of V with respect to (2.1) as,

$$D^+V(t,x) = \limsup_{h \to 0^+} \tfrac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)] \qquad (2.2)$$

Note that in (2.1), $D^+V(t,x)$ is a functional whereas V is a function.

Definition 2.2.

A function $g(t,u)$ is said to be quasimonotone nondecreasing in u, if $u \le v$ and $u_i = v_i$ for $1 \le i \le N$ implies $g_i(t,u) \le g_i(t,v)$ for all i.

Definition 2.3.

The trivial solution $x = 0$ of (2.1) is said to be

(ES_1) eventually stable if for every $\varepsilon > 0$ *and* $t_0 \in R_+$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ such that for any $x_0 \in R^n$ the inequality $\|x_0\| < \delta$ implies $\|x(t,t_0,x_0)\| < \varepsilon$ *for* $t \ge t_0$;

(ES_2) uniformly eventually stable if for every $\varepsilon > 0$ *and* $t_0 \in R_+$ there exist $\delta = \delta(\varepsilon) > 0$ such that for any $x_0 \in R^n$, the inequality $\|x_0\| < \delta$ implies $\|x(t,t_0,x_0)\| < \varepsilon$ *for* $t \ge t_0$;

(ES_3) asymptotically eventually stable if it is stable and if for each $\varepsilon > 0$ *and* $t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon)$ such that for $t \ge t_0 + T$ and $\|x_0\| \le \delta$ we have $\|x(t,t_0,x_0)\| < \varepsilon$.

(ES_4) uniformly asymptotically eventually stable if it is uniformly stable and $\delta_0 = \delta_0(\varepsilon)$ and $T = T(\varepsilon)$ such that for $t \ge t_0 + T$, the inequality $\|x_0\| \le \delta$ implies $\|x(t,t_0,x_0)\| < \varepsilon$.

Definition 2.4.

A function $a(r)$ is said to belong to the class K if $a \in C[[0,\rho), R_+], a(0) = 0,$ *and* $a(r)$ is strictly monotone increasing in r.

In this paper, we define the following sets:

$$\bar{S}_\psi = \{x \in R^N : \|x\| \le \psi\}$$

$$S_\psi = \{x \in R^N : \|x\| < \psi\}$$

Remark 2.2.

The inequalities between vectors are understood to be component-wise inequalities.

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Definition 2.5.

A function $b(r)$ is said to belong to a class L if $b \in C[J, R_+]$, $b(t)$ is monotone decreasing in t *and* $b(t) \to 0$ *as* $t \to \infty$.

Definition 2.6.

A function $a(t,r)$ is said to belong to the class KK if $a \in C[[0,\rho), R_+], a \in K$ for each $t \in J$, and a is monotone increasing in t for each $r > 0$ and $a(t,r) \to \infty$ *as* $t \to \infty$ for each $r > 0$.

Definition 2.7.

A function $V(t,x)$ with $V(t,0) = 0$ is said to be positive definite if there exists a function $a \in K$ such that the relation $V(t,x) \geq a(\|x\|)$ is satisfied for $(t,x) \in J \times S_\rho$.

Definition 2.8.

A function $V(t,x)$ with $V(t,0) = 0$ is said to be negative definite if there exists a function $a \in K$ such that the relation $V(t,x) \leq -a(\|x\|)$ is satisfied for $(t,x) \in J \times S_\rho$.

Definition 2.9.

A function $V(t,x) \geq 0$ is said to be decrescent if there exists a function $a \in K$ such that the relation $V(t,x) \leq a(\|x\|)$ is satisfied for $(t,x) \in J \times S_\rho$.

Alongside (2.1), we shall consider a comparison system of the form

$$\begin{gathered} u' = g(t,u), t \neq t_k,\ t \geq t_0, k = 1,2,... \\ \Delta u = \psi_k(u(t_k)),\ t = t_k, \\ u(t_0^+) = u_0 \end{gathered} \tag{2.3}$$

existing for $t \geq t_0$, where $u \in R^n$, elation $V(t,x) \leq a(\|x\|)$ is satisfied for $(t,x) \in J \times S_\rho$. existing for $t \geq t_0$, $u \in R^n$, $R_+ = [t_0, \infty)$, $g : R_+ \times R_+^n \to R^n$, $g(t,0) \equiv 0$, where g is the continuous mapping of $R_+ \times R_+^n$ into R^n. The function $g \in PC[R_+ \times R_+^n, R^n]$ is such that for any initial data $(t_0, u_0) \in R_+ \times R^n$, the system (2.3) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t, t_0, u_0) \in PC([t_0, \infty), R^n)$. Note that some sufficient conditions for the existence of solution of (2.3) has been examined in [9,18, 24, 26].

Lemma 2.1. Assume that the hypotheses $A_0(i), (ii), (iii)$ hold, and that $f(t,0) \equiv 0$ and that $I_k(0) \equiv 0$. Then the interval J can be extended to the maximal interval of existence $[t_0, \infty)$.

Proof.

Since the conditions $A_0(i), (ii), (iii)$ hold, and that $f(t,0) \equiv 0$ and that $I_k(0) \equiv 0$, then from the existence theorem for the equation $x' = f(t, x(t))$ [18] with impulses, it follows that the

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solution $x(t) = x(t, t_0, x_0)$ of the IVP (2.1) is defined on each of the intervals $(t_{k-1,}\ t_k],\ k = 1,2,...$ Again, since $0 < t_1 < t_2 < ... and \lim_{k\to\infty} t_k = \infty,$ then we conclude that the interval J can be continued to $[t_0, \infty)$ for t_0.

MAIN RESULTS

In this section we begin by proving the comparison results, then proceed to establish the necessary conditions for the eventual stability of the set $x = 0$ of the impulsive differential systems with fixed moments of impulse.

Using (2.3), Definition 2.2 can be analogously defined as follows:

Definition 3.1.

The trivial solution $u = 0$ of (2.3) is said to be

(ES_1^*) eventually stable if for every $\varepsilon > 0\ and\ t_0 \in R_+$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ such that for any $x_0 \in R^n$ the inequality $\|u_0\| < \delta$ implies $\|u(t, t_0, u_0)\| < \varepsilon\ for\ t \geq t_0$;

(ES_2^*) uniformly eventually stable if the δ in (ES_1^*) above is independent of t_0 i.e. for every $\varepsilon > 0\ and\ t_0 \in R_+$ there exist $\delta = \delta(\varepsilon) > 0$ such that the inequality $\|u_0\| < \delta$ implies $\|u(t, t_0, u_0)\| < \varepsilon\ for\ t \geq t_0$;

(ES_3^*) equiasymptotically eventually stable if ES_1^* is satisfied and given $\varepsilon > 0\ and\ t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon) > 0$ such that for $t \geq t_0 + T$ and $\|u_0\| \leq \delta$ we have $\|u(t, t_0, u_0)\| < \varepsilon,\ t \geq t_0 + T$.

(ES_4^*) uniformly asymptotically eventually stable if (ES_2^*) is satisfied (ES_3^*) is independent of t_0.

THEOREM 3.1. (Comparison results) Assume that

(i) $g \in C[J \times R_+^n, R^n],\ g(t,0) \equiv 0,\ and\ g(t,u)$ is quasimonotone nondecreasing in u for each $u \in R^n$ and $\lim_{(t,y)\to(t_k^+,u)} g(t,u) = g(t_k^+, u)$ exists;

(ii) $r(t) = r(t, t_0, u_0) \in PC([t_0, T), R^n)$ is the maximal solution of (2.3) existing for $t \geq t_0$.

(iii) $V \in C[J \times S_\psi, R_+^N],\ V \in L\ such\ that\ for\ (t,x) \in J \times S_\psi,$

$D^+V(t,x) \leq g(t, V(t,x)),\ t \neq t_k$

and

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$V(t, x + I_k(x(t_k)) \le \rho_k(V(t,x)),\ t = t_k,\ x \in S_\psi$ and the function $\rho_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

(iv) $x(t) = x(t, t_0, x_0) \in PC([t_0, T], R^N)$ is a solution of (2.1) such that

$$V(t_0, x_0) \le u_0 \qquad (3.1)$$

existing for $t \ge t_0$. Then

$$V(t, x(t)) \le r(t) \qquad (3.2)$$

Proof.

Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1) existing for $t \ge t_0$, such that $V(t_0, x_0) \le u_0$.

Set $m(t) = V(t, x(t))$ for $t \ne t_k$ so that for small $h > 0$, using (2.2) we have

$$m(t+h) - m(t) = V(t+h, x(t+h)) - V(t+h, x(t) + hf(t, x(t)) + V(t+h, x(t) + hf(t, x(t)) - V(t, x)$$

Since $V(t,x)$ is locally Lipschitzian in x for $t \in [t_{0,}\ \infty)$, we have

$$m(t+h) - m(t) \le k\|x(t+h) - (x(t) + hf(t, x(t)))\| + V(t+h, x(t) + hf(t, x(t)) - V(t, x)$$

Dividing through by $h > 0$, and taking the $\limsup$ as $h \to 0^+$ we have

$$\limsup_{h\to 0^+} \tfrac{1}{h}[m(t+h) - m(t)] \le \limsup_{h\to 0^+} \tfrac{i}{h}\left[k\|x(t+h) - x(t) - hf(t, x(t)\|e\right] + \limsup_{h\to 0^+} \tfrac{1}{h}[V(t+h, x(t) + hf(t, x(t)) - V(t, x)]$$

where k is the local Lipschitz constant and $e = (1,1,...,1)^T$.

It follows that

$$D^+ m(t) = D^+ V(t, x(t)) \le g(t, m(t)),$$

and using condition (ii) of Theorem 3.1 we arrive at

$$V(t, x(t)) \le r(t),$$

provided

$V(t_0, x_0) \le u_0$.

Also,

$$m(t_k^+) = V(t_k^+, x(t_k) + I_k(x(t_k^+)) \le \psi_k(m(t_k^+)).$$

Hence, by Cor. 1.7.1 in [13], we obtain the desired estimate of (3.1).

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Corollary 3.2. Assume that:

(i) $g \in C[J \times R_+^n, R^n]$, $g(t,0) \equiv 0$, *and* $g(t,u)$ is quasimonotone nondecreasing in u for each ` $u \in R^n$ and $\lim_{(t,y)\to(t_k^+,u)} g(t,u) = g(t_k^+,u)$ exists;

(ii) $p(t) = p(t,t_0,u_0) \in PC([t_0,T), R^n)$ is the minimal solution of (2.3) existing for $t \geq t_0$.

(iii) $V \in C[J \times S_\psi, R_+^N]$, $V \in L$ *such that for* $(t,x) \in J \times S_\psi$,

$D^+V(t,x) \geq g(t,V(t,x)),\ t \neq t_k$

and

$V(t, x + I_k(x(t_k)) \geq \rho_k(V(t,x)),\ t = t_k,\ x \in S_\psi$ and the function $\rho_k : R_+^N \to R_+^N$ is

nondecreasing for $k = 1,2,\ldots$

(iv) $x(t) = x(t,t_0,x_0) \in PC([t_0,T], R^N)$ is a solution of (2.1) such that

$$V(t_0,x_0) \geq u_0 \quad (3.3)$$

existing for $t \geq t_0$. Then

$$V(t,x(t)) \geq p(t) \quad (3.4)$$

Proof.

Let $x(t) = x(t,t_0,x_0)$ be any solution of (2.1) existing for $t \geq t_0$, such that $V(t_0,x_0) \geq u_0$.

Set $m(t) = V(t,x(t))$ for $t \neq t_k$ so that for small $h > 0$, using (2.2) we have

$$m(t+h) - m(t) = V(t+h, x(t+h)) - V(t+h, x(t) + hf(t,x(t)) + V(t+h, x(t) + hf(t,x(t)) - V(t,x)$$

Since $V(t,x)$ is locally Lipschitzian in x for $t \in [t_{0,}\ \infty)$, we have

$$m(t+h) - m(t) \geq k\|x(t+h) - (x(t) + hf(t,x(t)))\| + V(t+h, x(t) + hf(t,x(t)) - V(t,x)$$

Dividing through by $h > 0$, and taking the $\limsup$ as $h \to 0^+$ we have

$$\limsup_{h\to 0^+} \tfrac{1}{h}[m(t+h) - m(t)] \geq \limsup_{h\to 0^+} \tfrac{i}{h}\Big[k\|x(t+h) - x(t) - hf(t,x(t)\|\Big]e + \limsup_{h\to 0^+} \tfrac{1}{h}[V(t+h, x(t) + hf(t,x(t)) - V(t,x)]$$

where k is the local Lipschitz constant and $e = (1,1,\ldots,1)^T$

It follows from condition (ii) of Cor 3.2 we arrive atat the estimate

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$$D^+m(t) = D^+V(t, x(t)) \geq g(t, m(t)), t \neq t_k,\ m(t_0^+) \geq u_0 \qquad (3.5)$$

Also,

$$m(t_k^+) = V(t_k^+, x(t_k) + I_k(x(t_k^+)) \geq \psi_k(m(t_k^+)) \qquad (3.6)$$

Hence, by Cor. 1.7.1 in [13], we obtain the desired estimate of (3.5).

In what follows, we shall obtain sufficient conditions for the eventual stability of the system (2.3).

THEOREM 3.2. Assume that:

(i) $g \in C[J \times R_+^n, R^n],\ g(t,0) \equiv 0,\ and\ g(t,u)$ is quasimonotone nondecreasing in u for fixed $t \in J$.

(ii) $V \in C[J \times S_\rho, R_+^n], V(t,x)$ is locally Lipschitzian in x and $\sum_{i=1}^{N} V_i(t,x) \to 0\ as\ \|x\| \to 0$ for each t and $(t,x) \in J \times S_\rho$,

$$D^+V(t,x) \leq g(t, V(t,x)) \qquad (3.7)$$

(iii) $for\ (t,x) \in J \times S_\rho$,

$$b(\|x(t)\|) \leq \sum_{i=1}^{N} V_i(t,x) \leq a(t, \|x\|) \qquad (3.8)$$

where $b \in K, a(t,.) \in K$ whence $a \in C[J \times S_\rho, R_+]$

Then the eventual stability of the set of trivial solution $u = 0$ of the system (2.3) implies the eventual stability of the set of trivial solution $x = 0$ of the system (2.1).

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given.

Assume that the solution (2.3) is eventually stable. Then given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_o, \varepsilon) > 0$ which is continuous in t_0 for each $\in$ such that

$$\sum_{i=1}^{N} u_{i0} \leq \delta \text{ implies } \sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\in), t \geq t_0 \qquad (3.9)$$

Since $V(t,x)$ is continuous and mildly unbounded i.e. $V(t,0) \to 0\ as\ \|x\| \to 0,$ then by the property of continuity, given $\varepsilon > 0$ there exists a positive function $\delta_1 = \delta_1(t_o, \varepsilon) > 0$ that is continuous in t_o for each $\in$ such that the inequalities

$$\|x_0\| < \delta_1\ \ and\ \ V(t_0, x_0) < \delta \qquad (3.10)$$

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are satisfied simultaneously.

We claim that if $\|x_0\| < \delta_1$ then $\|x(t,t_0,x_0)\| <\in$ by the stability of $x(t)$.

Now suppose this claim is false, the there would exists a point $t_1 \in [t_0,t)$ and the solution $x(t,t_0,x_0)$ with $\|x_0\| < \delta_1$ such that

$$\|x(t_1)\| = \varepsilon \ \ and \ \ \|x(t)\| < \varepsilon \text{ for } t \in [t_0,t_1) \tag{3.11}$$

So that using equation (3.11); (3.8) reduces to the form

$$b(\|x(t_1)\|) \leq \sum_{i=1}^{N} V_i(t_1,x(t_1)) \text{, implying}$$

$$b(\varepsilon) \leq \sum_{i=1}^{N} V_i(t_1,x(t_1)) \tag{3.12}$$

This implies that $x(t) \in S_\psi$ for $t \in [t_0,t_1)$ and from Theorem 3.1,

$$V(t,x(t)) \leq r(t,t_0,u_0), \tag{3.13}$$

where $r(t,t_0,u_0)$ is the maximal solution of (3.2.3).

Then using equations (3.8), (3.9), (3.12) and (3.13) we arrive at the estimate

$$b(\varepsilon) \leq V_0(t_1,x(t_1) \leq \sum_{i=1}^{N} r_i(t,t_0,u_0) < b(\varepsilon)$$

which leads to a contradiction.

Hence, the eventual stability of the set of trivial solution $u = 0$ of (2.3) implies the corresponding eventual stability of the set of trivial solution $x = 0$ of (2.1).

In what follows, we shall establish sufficient conditions for the asymptotic stability of the main system (2.1).

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THEOREM 3.3. Assume that:

(i) $g \in C[J \times R_+^n, R^n]$, $g(t,0) \equiv 0$, *and* $g(t,u)$ is quasimonotone nondecreasing in u for fixed $t \in J$.

(ii) $V \in C[J \times S_\rho, R_+^n], V(t,x)$ is locally Lipschitzian in x and $\sum_{i=1}^{N} V_i(t,x) \to 0 \; as \; \|x\| \to 0$ for each t and $(t,x) \in J \times S_\rho$,

$$D^+V(t,x) \le g(t, V(t,x)) \tag{3.7}$$

(iii) *for* $(t,x) \in J \times S_\rho$,

$$b(\|x(t)\|) \le \sum_{i=1}^{N} V_i(t,x) \le a(t, \|x\|) \tag{3.8}$$

where $b \in K$, $a(t,.) \in K$ whence $a \in C[J \times S_\rho, R_+]$

Then the asymptotic eventual stability of the set of trivial solution $u = 0$ of the system (2.3) implies the asymptotic eventual stability of the set of trivial solution $x = 0$ of the system (2.1).

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given.

Assume that the solution (2.3) is eventually stable. Then given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_o, \varepsilon) > 0$ which is continuous in t_0 for each $\in$ such that

$$\sum_{i=1}^{N} u_{i0} \le \delta \text{ implies } \sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\in), t \ge t_0 \tag{3.9}$$

Since $V(t,x)$ is continuous and mildly unbounded i.e. $V(t,0) \to 0 \; as \; \|x\| \to 0,$ then by the property of continuity, given $\varepsilon > 0$ there exists a positive function $\delta_1 = \delta_1(t_o, \varepsilon) > 0$ that is continuous in t_o for each $\in$ such that the inequalities

$$\|x_0\| < \delta_1 \;\; and \;\; V(t_0, x_0) < \delta \tag{3.10}$$

are satisfied simultaneously.

We claim that if $\|x_0\| < \delta_1$ then $\|x(t, t_0, x_0)\| < \in$ by the stability of $x(t)$.

Now suppose this claim is false, the there would exists a point $t_1 \in [t_0, t)$ and the solution $x(t, t_0, x_0)$ with $\|x_0\| < \delta_1$ such that

$$\|x(t_1)\| = \varepsilon \;\; and \;\; \|x(t)\| < \varepsilon \text{ for } t \in [t_0, t_1) \tag{3.11}$$

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So that using equation (3.11); (3.8) reduces to the form

$$b(\|x(t_1)\|) \leq \sum_{i=1}^{N} V_i(t_1, x(t_1)) \text{, implying}$$

$$b(\varepsilon) \leq \sum_{i=1}^{N} V_i(t_1, x(t_1)) \qquad (3.12)$$

This implies that $x(t) \in S_\psi$ for $t \in [t_0, t_1)$ and from Theorem 3.1,

$$V(t, x(t)) \leq r(t, t_0, u_0), \qquad (3.13)$$

where $r(t, t_0, u_0)$ is the maximal solution of (3.2.3).

Then using equations (3.8), (3.9), (3.12) and (3.13) we arrive at the estimate

$$b(\varepsilon) \leq V_0(t_1, x(t_1) \leq \sum_{i=1}^{N} r_i(t, t_0, u_0) < b(\varepsilon)$$

which leads to an absurdity.

Hence, the asymptotic eventual stability of the set of trivial solution $u = 0$ of (2.3) implies the asymptotic eventual stability of the set of trivial solution $x = 0$ of (2.1).

CONCLUSION

This paper examined the existence of maximal solution of the comparison system via comparison principle as well as sufficient conditions for Lyapunov asymptotic eventual stability of nonlinear impulsive differential equations using the vector Lyapunov functions. Herein, it was established that, the maximal solution of the comparison system majorizes the vector form of the Lyapunov functions. By splitting the Lyapunov function into components, each of the solution states or the solution vectors can be inserted into each of the components of the Lyapunov function rather than in the whole Lyapunov function. This way, it becomes easier for the Lyapunov function to adequately predicts the behavior of the solution vectors so that the “restrictions and difficulties” imposed on the Lyapunov function which is encountered by the use of the scalar Lyapunov function is reduced. Together with comparison results, sufficient conditions for the asymptotic eventual stability of the impulsive differential systems are presented.

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