



## DEFLECTION OF AN ISOTROPIC ALL-ROUND SIMPLY SUPPORTED RECTANGULAR PLATES INCORPORATING SHEAR EFFECT USING CHARACTERISTIC ORTHOGONAL POLYNOMIAL FUNCTION

Sopakirite Somina<sup>1</sup> and Happiness D. Mac-Eteli<sup>2</sup>

<sup>1</sup>Department of Civil Engineering, Federal University Otuoke, Bayelsa State, Nigeria

<sup>2</sup>Department of Civil Engineering, Niger Delta University, Bayelsa State, Nigeria

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**ABSTRACT:** *The breadth and dexterity with which plates are used in the vast majority of engineering structures necessitates an ever-increasing and deeper study focus on plate strength and stiffness at the ultimate and serviceability limit states of response. From Kirchhoff's hypothesis, thin plates when subjected to transverse loading, bend and experience transverse deflections that are typically minor in comparison to the plate thickness. However, for thicker plates there is an observed limitation in the application of Kirchhoff's hypothesis, as this theory ignore the effect of transverse shear on the deformation of plates. This study therefore analytically examines the effect of induced shear on the deflection indices of plates with varying aspect ratios, using the characteristic orthogonal polynomial function. Result obtained shows a close agreement between present study and Kirchhoff's hypothesis for membrane and thin plates. However, a significant difference was observed for moderately thick plates and thick plates, which clearly shows the effect of transverse shear as the plate thickness increases, which further validates the limitations of Kirchhoff's hypothesis for moderately thick as well as thick plates. For an aspect ratio of 1.0 – 2.0 at 0.1 interval, results obtained indicated a percentage difference in deflection between the Present study and Kirchhoff's hypothesis to range between - 0.040 – 3.508%, 0.527 – 3.552%, 4.266 – 5.858%, and 13.980 – 17.011% for membrane, stiffened, moderately thick, and thick plates respectively. The validation of the Kirchhoff's hypothesis for membrane as well as stiffened plates by the present study, indicates the suitability of the application of the characteristic orthogonal polynomial function in the evaluation of the deflection of plates regardless of thickness.*

**KEY WORDS:** Kirchhoff's Hypothesis, Plates, Deflection, Orthogonal, Polynomial Function, Shear Effect



## INTRODUCTION

Plates are straight, plane, two-dimensional structural components of which one dimension, referred to as thickness,  $h$ , is much smaller than other dimensions. Bending and transverse shear occur when a plate is exposed to a stress perpendicular to its plane. Plates are beam generalizations (Blaauwendraad, 2010), however, a beam can only span one way, whereas a plate can transport loads in both directions. Geometrically, they are bounded either by straight or curve lines. Like their counterparts, the beams, they are not only serve as structural components but can also form complete structures such as slab bridges for example. Statically, plates have free, simply supported and fixed boundary conditions, including elastic supports and elastic restraints or, in some cases, even point supports.

The extend and dexterity of application of plates in vast majority of engineering structures call for ever increase and deeper research interest in the strength and stiffness characteristics of plate at the ultimate and serviceability state of response. In 1876, Kirchhoff (1876) published an important thesis on thin plates. In this work, he stated two independent basic assumptions that are now widely accepted in the plate-bending theory and are known as “Kirchhoff’s hypothesis”. This hypothesis permitted the creation of the classical bending theory of thin plates which for more than a century has been the basis for the calculation and design of structures in various areas of engineering and has yielded important theoretical and numerical results. However, Kirchhoff’s theory (Kirchhoff, 1876) had some drawbacks and deficiencies which is related to the neglects of the deformation caused by transverse shear, hence, lead to considerable errors if applied to moderately thick and thick plates. For such plates, Kirchhoff’s classical theory under-estimates deflections and over-estimates frequencies and buckling loads. While considering the effect of shear stresses, several solutions for bending and buckling of beams and plates were offered by Lokshin et al. (2009).

Several researchers have endeavored to improve Kirchhoff’s theory and such attempts continue to this day. The most important advance in this direction was made by (Senjanović et al 2013, Shimpi et al 2018). Their theory takes into account the influence of the transverse shear deformation on the deflection of the plate and leads to a sixth-order system of governing differential equations, and accordingly, to three boundary conditions on the plate edge. Here it is unnecessary to introduce the effective transverse shear force. The theory of (Senjanović et al 2013, Shimpi et al 2018) is free from the drawbacks of Kirchhoff’s theory (Kirchhoff, 1876). Later, Mindlin (1951) developed first order shear deformation theory considering the effect of transverse shear deformation in the analysis of plates. However, this theory does not certify the shear stress condition at the top and bottom of the plate’s thickness and require a shear correction factor.

Other notable works on the refined plate theory includes: A formulation based on displacement approach was made by (Levinson, 1980) and his theory does not require shear correction factor, Also (Oguaghamba, 2015) evaluated the static analysis of an isotropic rectangular plate with various boundary conditions using direct variation method according to Ritz to obtain the total potential energy of plate, refined nonlinear shear deformation of thick rectangular plate was presented by (Ibearugbulem et al 2014) using a modified mixed variational formulation. The effects of plate thickness, charge mass, and confinement degree on the dynamic response of square plates were explored by Geretto et al. (2015) who conducted experiments to investigate the plastic deformations of square plates subjected to fully confined blast loading.



For all practical purposes, three-dimensional effects, such as the influence of plate thickness on stress components, are typically neglected or dismissed (Kotousov 2010). In general, the case of relatively thin and moderately thick plates, including shear deformation in plate deflection is advised (Zietlow et al, 2012).

## MATERIALS AND METHODS

### Materials

The research is related to rectangular plate made from isotropic, homogenous and elastic materials with constant thickness and yield stress. The constants include, elastic moduli in x, y and xy directions. The plate is all round simply supported with a uniformly distributed load over the plate's area.

### Characteristics Orthogonal Polynomials Function.

The assumed deflection shape of plate normally formulated by inspection and sometimes by trial and error until Bhat (Bhat, 1958) proposed a systematic method of constructing such shape functions in the form of Characteristic Orthogonal Polynomial (COPs). According to Bhat, the stress function for a rectangular plate is assumed to be the product of two independent functions; one of which is a pure function of x and the other is a function of y such that:

$$\phi(x, y) = F(x) \cdot G(y) = \sum_{m=0}^{\infty} X_m x^m Y_n y^n \quad (1)$$

Expressing Equation (1) in the form of non-dimensional parameters, say R and Q for x and y axis respectively, we have:

$$\phi(R, Q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_m B_n R^m Q^n \quad (2)$$

Where,

$$A_m = X_m a^m, \quad B_n = Y_n b^n$$

For a beam with an arbitrary support condition subjected to uniformly distributed load (UDL) along an arbitrary direction, it can be seen that due to this applied load, reactive forces such as, moments and reactions will develop at its support and the deflection function for such a beam will be a fourth order function. This suggest that the polynomial of Equation (2) is a fourth order function. Therefore, expanding Equation (2) to 4<sup>th</sup> series yields:

$$\phi(R, Q) = (A_0 + A_1 R + A_2 R^2 + A_3 R^3 + A_4 R^4)(B_0 + B_1 Q + B_2 Q^2 + B_3 Q^3 + B_4 Q^4) \quad (3)$$

Equation (3) represents the general stress function for rectangular plates.

Where the coefficients  $A_m$  and  $B_n$  of the series are determined from the boundary conditions at the edges of the plate.



## THE GOVERNING EQUATIONS OF THE REFINED PLATE BENDING THEORY

The governing differential equation of the plate having the effect of transverse shear was given by Vesil'ev (1998) as follows:

$$D\nabla^2\nabla^2\phi = P \quad (4a)$$

$$\nabla^2\Psi - K^2\Psi = 0 \quad (4b)$$

And the deflection of the plate is expressed as:

$$w = \phi - \frac{D}{C}\nabla^2\phi \quad (5)$$

Where;

P is the transverse load on the plate,  $\phi$  is the stress function of the plate, and  $\Psi$  is the stream function.

$$K^2 = \frac{2C}{D(1-\mu)} \quad (6)$$

D is the flexural rigidity of the plate, and it is given as:

$$D = \frac{Eh^3}{12(1-\mu^2)} \quad (7)$$

And C describes the shear stiffness of the plate in the planes xz and yz and it is expressed as:

$$C = Gh \quad (8)$$

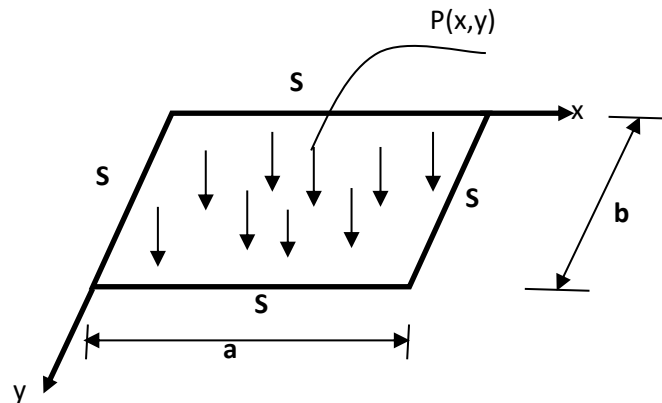
Where

$$G = \frac{E}{2(1+\mu)} \quad (9)$$

Where  $\mu$ , h, and E are Poisson's ratio, thickness and Young's modulus of elasticity of the plate respectively.

### Application of the Refined Theory to a SSSS Plate using COPs

Consider a simply supported rectangular plate with sides a and b subjected to uniform distributed load of intensity P(x,y) as shown in Figure 2 below:



**Figure 2: SSSS Rectangular plate**

The stress function was obtained from equation (3) using the boundary conditions of SSSS plate by Oguaghamba (2015) as:

$$\phi(R, Q) = A(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4) \quad (10)$$

Where  $A$  is the amplitude of the load,  $R$  and  $Q$  are non-dimensional terms in  $x$  and  $y$  directions respectively.

Substituting Equation (10) into Equation (4a) yields:

$$\left[ \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right] \left[ \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right] [A(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)] = \frac{P}{D} \quad (11)$$

Expressing Equation (11) in non-dimensional terms, where  $x = aR$ ,  $y = bQ$ ,  $dx = adR$ , and  $dy = bdQ$ , for  $0 \leq R \leq 1$ ;  $0 \leq Q \leq 1$ . Yields:

$$\left[ \frac{d^2}{a^2 dR^2} + \frac{d^2}{b^2 dQ^2} \right] \left[ \frac{d^2}{a^2 dR^2} + \frac{d^2}{b^2 dQ^2} \right] [A(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)] = \frac{P}{D} \quad (12)$$

Expanding and simplifying Equation (12) and substituting  $a = rb$ , yields:

$$A \left[ \frac{24}{r^4 b^4} (Q - 2Q^3 + Q^4) + \frac{2}{r^2 b^4} (-12R + 12R^2)(-12Q + 12Q^2) + \frac{24}{b^4} (R - 2R^3 + R^4) \right] = \frac{P}{D}$$

(13)

Where the aspect ratio  $r = \frac{a}{b}$

Therefore, from Equation (13), the amplitude of the load  $A$  is:



$$A = \frac{Pb^4}{D \left[ \frac{24}{r^4}(Q-2Q^3+Q^4) + \frac{2}{r^2}(-12R+12R^2)(-12Q+12Q^2) + 24(R-2R^3+R^4) \right]} \quad (14)$$

Also substituting Equation (10) into Equation (5) yields:

$$w = A(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4) - \frac{D}{c} \left[ \frac{d^2}{a^2 dR^2} + \frac{d^2}{b^2 dQ^2} \right] [A(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)] \quad (15)$$

Simplifying Equation (15) and substituting  $a = rb$ , gives:

$$w = A \left[ (R - 2R^3 + R^4)(Q - 2Q^3 + Q^4) - \frac{D}{b^2 c} \left[ \frac{1}{r^2} (-12R + 12R^2)(Q - 2Q^3 + Q^4) + (R - 2R^3 + 12R^4)(-12Q + Q^2) \right] \right] \quad (16)$$

Substituting Equation (14) into Equation (16), yields the equation of the deflected surface of an all-round simply supported rectangular plate having shear effect.

$$w = \left[ \frac{Pb^4}{D \left[ \frac{24}{r^4}(Q-2Q^3+Q^4) + \frac{2}{r^2}(-12R+12R^2)(-12Q+12Q^2) + 24(R-2R^3+R^4) \right]} \right] \left[ (R - 2R^3 + R^4)(Q - 2Q^3 + Q^4) - \frac{D}{b^2 c} \left[ \frac{1}{r^2} (-12R + 12R^2)(Q - 2Q^3 + Q^4) + (R - 2R^3 + 12R^4)(-12Q + Q^2) \right] \right] \quad (17)$$

Where at maximum deflection,  $R = Q = 0.5$

Substituting the above values of R and Q into Equation (17), yields:

$$w_{max} = \left[ \frac{Pb^4}{D \left( \frac{7.5}{r^4} + \frac{18}{r^2} + 7.5 \right)} \right] \left[ 0.0977 - \frac{D}{b^2 c} \left( -\frac{0.9375}{r^2} - 0.9375 \right) \right] \quad (18)$$

Only the first four series of the shape function of the deflected surface  $w(x,y)$  affects the maximum deflection utilizing orthogonal polynomial; thus, a numeric constant (1.4), indicating a 40% increase is employed for linearizing the effect of minor variations, thereby yielding the maximum deflection of plates as presented by the present study and shown in equation 19 below;

$$w_{max} = 1.4 * \left[ \frac{Pb^4}{D \left( \frac{7.5}{r^4} + \frac{18}{r^2} + 7.5 \right)} \right] \left[ 0.0977 - \frac{D}{b^2 c} \left( -\frac{0.9375}{r^2} - 0.9375 \right) \right] \quad (19)$$



## RESULTS AND DISCUSSION

### Results

The characteristic orthogonal polynomial function has been used to evaluate the maximum deflection in different types of plates using the following physical and geometric properties:  $b = 4\text{m}$ , Aspect Ratio ( $r$ ) = 1 – 2, at 0.1m interval, load ( $P$ ) = 150KN, Young's Modulus of Elasticity ( $E$ ) = 205MPa, Poison's Ratio ( $\mu$ ) = 0.3. Results obtained are therefore as shown below;

**Table 1: Result for the deflection of membrane plates**

b	Aspect Ratio, r	Maximum Deflection (W) in m		Diff (%)
		Present Study	Classical Method	
4	1	0.0679	0.0679	-0.040
4	1.1	0.0815	0.0814	0.044
4	1.2	0.0948	0.0946	0.244
4	1.3	0.1078	0.1072	0.547
4	1.4	0.1202	0.1191	0.899
4	1.5	0.1319	0.1302	1.282
4	1.6	0.1429	0.1405	1.680
4	1.7	0.1531	0.1499	2.084
4	1.8	0.1626	0.1586	2.472
4	1.9	0.1714	0.1666	2.852
4	2	0.1796	0.1733	3.508

**Table 2: Result for the deflection of stiffened plates**

b	Aspect Ratio, r	Maximum Deflection (W) in m		Diff (%)
		Present Study	Classical Method	
4	1	0.0025	0.0025	0.527
4	1.1	0.0030	0.0030	0.556
4	1.2	0.0035	0.0035	0.724
4	1.3	0.0040	0.0040	0.994
4	1.4	0.0045	0.0044	1.325
4	1.5	0.0049	0.0048	1.690
4	1.6	0.0053	0.0052	2.073
4	1.7	0.0057	0.0056	2.459
4	1.8	0.0060	0.0059	2.836
4	1.9	0.0064	0.0062	3.202
4	2	0.0067	0.0064	3.552

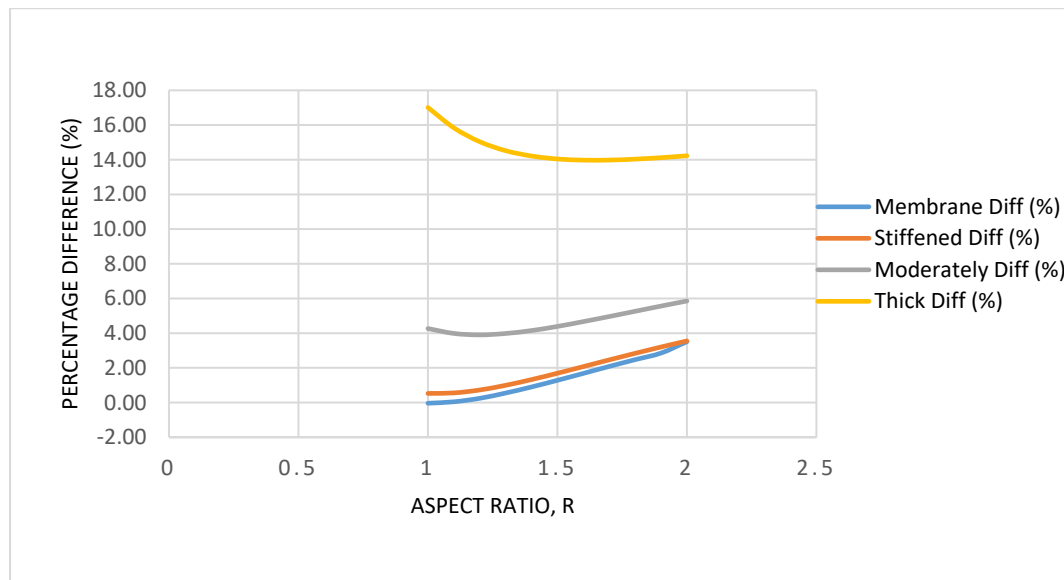
**Table 3: Result for the deflection of moderately thick plates**

b	Aspect Ratio, r	Maximum Deflection (W) in m		Diff (%)
		Present Study	Classical Method	
4	1	0.00013853	0.00013262	4.266
4	1.1	0.00016562	0.00015902	3.985
4	1.2	0.00019227	0.00018477	3.901
4	1.3	0.00021806	0.00020938	3.981
4	1.4	0.00024267	0.00023260	4.150
4	1.5	0.00026593	0.00025426	4.388
4	1.6	0.00028775	0.00027432	4.667
4	1.7	0.00030808	0.00029280	4.960
4	1.8	0.00032697	0.00030977	5.260
4	1.9	0.00034446	0.00032530	5.562
4	2	0.00036062	0.00033951	5.854

**Table 4: Result for the deflection of thick plates**

b	Aspect Ratio, r	Maximum Deflection (W) in m		Diff (%)
		Present Study	Classical Method	
4	1	0.00001665	0.00001382	17.011
4	1.1	0.00001969	0.00001657	15.840
4	1.2	0.00002266	0.00001926	15.038
4	1.3	0.00002553	0.00002182	14.521
4	1.4	0.00002826	0.00002424	14.213
4	1.5	0.00003083	0.00002650	14.043
4	1.6	0.00003323	0.00002859	13.977
4	1.7	0.00003547	0.00003051	13.980
4	1.8	0.00003755	0.00003228	14.034
4	1.9	0.00003947	0.00003390	14.118
4	2	0.00004125	0.00003538	14.225





**Figure 1: Effect of aspect ratio on percentage difference between deflections of plates from present study and Kirchhoff's hypothesis**

## DISCUSSION OF RESULTS

Maximum deflections for membrane, stiffened, moderately thick and thick plates were evaluated based on the above data and the result of the Present Study compared to that of the Classical Method (based on Kirchhoff's hypothesis). A close observation of these numerical solutions as shown in Tables 1 and 2 indicates clearly the validation of the classical method for evaluating the maximum deflections of membrane as well as stiffened plates. This is as a result of the minimal difference in numerical findings between the present method and the classical methods, ranging between -0.04 – 3.552. The results further indicate a polynomial relationship to the fourth order between the aspect ratio and the maximum deflections for all plate categories. This implies that the maximum deflection of plates is a function of the sum of multiple orders of the aspect ratio. This relationship was found to be statistically sound regardless of plate thickness. The change in maximum deflection (as represented by the percentage differences in the maximum deflections between the Present Study and the Classical method), was observed to be a polynomial function of the aspect ratio. This change is however deemed to be due to the induced shear effect, resulting from the transverse loading. For membrane as well as stiffened plates, increase in aspect ratio was observed to be directly proportional to the change in maximum deflection. The slope of the proportionality was observed to reduce consistently with increasing thickness, such that the proportionality was near inverse for thick plates (see figure 1). This implies that transversely induced shear effect on plates relative to aspect ratio is primarily a function of plate's thickness, which is highest at the least aspect ratio for thick plates, and lowest at the least aspect ratio for membrane and stiffened plates.



## CONCLUSION AND RECOMMENDATION

From findings of the study, the following conclusions and recommendations are drawn;

1. The neglected shear effect of transverse loads on the maximum deflection of membrane and stiffened plates by the Classical method is considered structurally adequate as no significant contribution was observed from the present study.
2. The application of the classical method for moderately thick and thick plates is not structurally ideal as the effect of shear as shown by the present study, yields significant contribution to maximum deflection.
3. Plate thickness is a primary factor relative to the contribution of shear to the maximum deflection of plates.
4. Change in maximum deflection due to induced shear effect is directly proportional to aspect ratio for membrane and stiffened plates but inversely proportional to aspect ratio (between 1 – 1.5) for thick plates.
5. The present study therefore recommends the integration of the characteristic orthogonal polynomial function in the evaluation of maximum deflection of plates regardless of thickness.

## REFERENCES

- Blaauwendraad, J. (2010). Thick Plates in Bending and Shear. In *Plates and FEM* (pp. 47-80). Springer, Dordrecht.
- Geretto, C.; Chung Kim Yuen, S.; Nurick, G.N. 2015. An experimental study of the effects of degrees of confinement on the response of square mild steel plates subjected to blast loading. *Int. J. Impact Eng.* 2015, 79, 32–44.
- Ibearugbulem, O. M., Ibeabuchi, V. T., & Njoku, K. O. (2014). Buckling analysis of SSSS stiffened rectangular isotropic plates using work principle approach. *International Journal of Innovative Research---amp---Development*, 3(11), 169-176.
- Kirchhoff, G. (1876). *Vorlesungen Uber Mathematische Physik*, B.G. Teubner, Leipzig
- Kotousov, A. (2010). Effect of plate thickness on stress state at sharp notches and the strength paradox of thick plates. *International Journal of Solids and Structures*, 47(14-15), 1916-1923.
- Levinson, M. (1980). An accurate, simple theory of the statics and dynamics of elastic plates. *Mechanics Research Communications*, 7(6), 343-350.
- Lokshin, A. Z., Mishkevich, V., & Ivanov, L. D. (2009). Effect of shear on deflection and buckling of beams and plates. *Ships and Offshore Structures*, 4(4), 323-336.
- Mindlin, R. D. (1951). Influence of Rotary Inertia and shear on flexural Motions of Isotropic Elastic Plates, *J. Appl. Mech.*, 18(1), 31-38.
- Oguaghamba, O. A. (2015). Analysis of Buckling and Post buckling Loads of Isotropic Thin Rectangular Plates. PhD Thesis: Department of Civil Engineering Federal University of Technology, Owerri Nigeria.
- Senjanović, I., Vladimir, N., & Tomić, M. (2013). An advanced theory of moderately thick plate vibrations. *Journal of Sound and Vibration*, 332(7), 1868-1880.



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- Shimpi, R. P., Guruprasad, P. J., & Pakhare, K. S. (2018). Single variable new first-order shear deformation theory for isotropic plates. *Latin American Journal of Solids and Structures*, 15(10).
- Zietlow D.W, Griffin D.C, Moore T.R., 2012. The limitations on applying classical thin plates clamped on the inner boundary, *AIP ADVANCES*.2012; 2: 1–8.
- Bhat, R. B. (1985). Natural frequencies of rectangular plates using characteristic orthogonal polynomials in Rayleigh-Ritz method. *Journal of Sound and Vibration*, 102(4), 493-499.
- Vasil'ev, V. V. (1998). Classical theory of plates: historical perspective and state-of-the-art. *Mechanics of solids*, 33(3), 35-45.

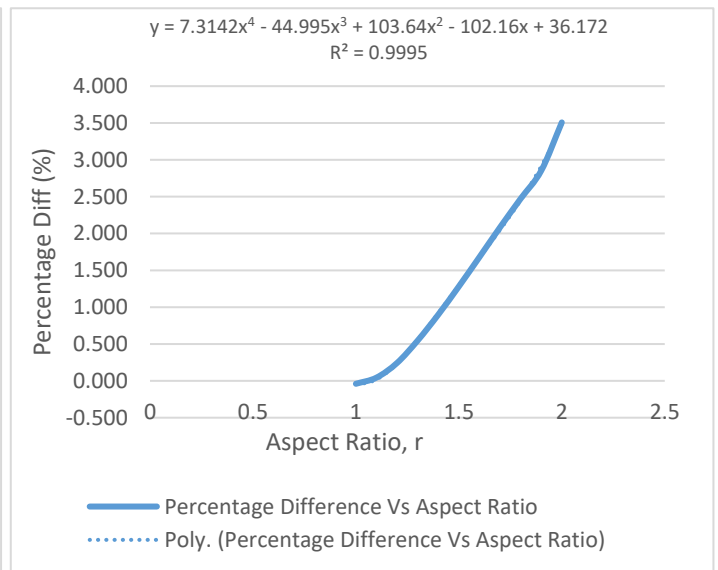
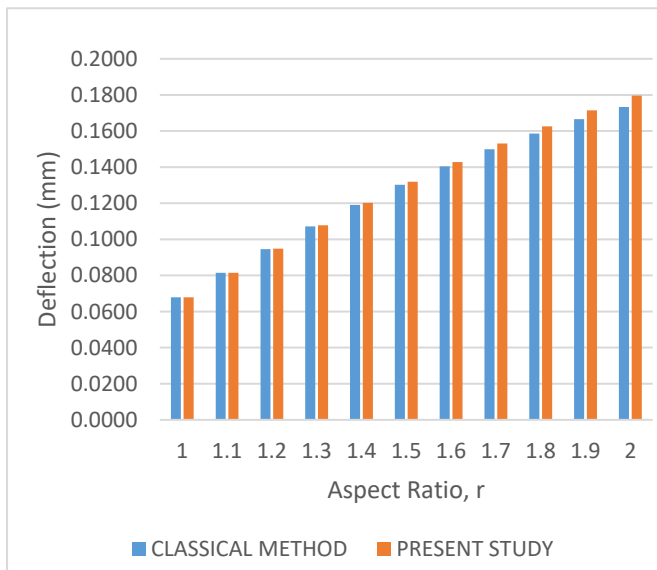


## APPENDICES

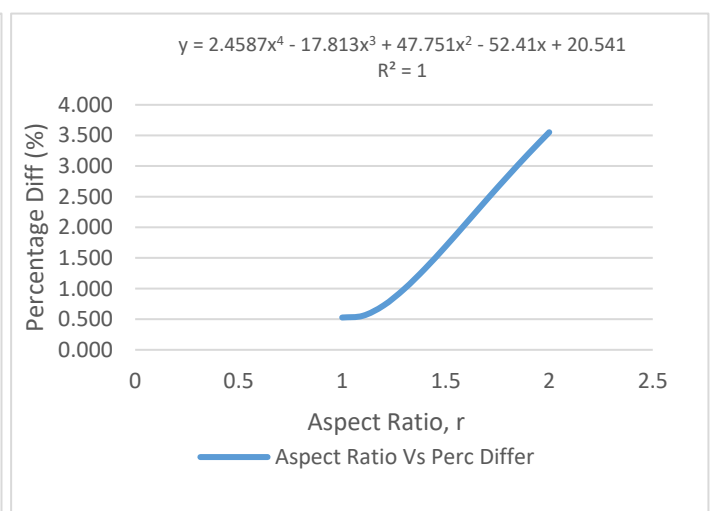
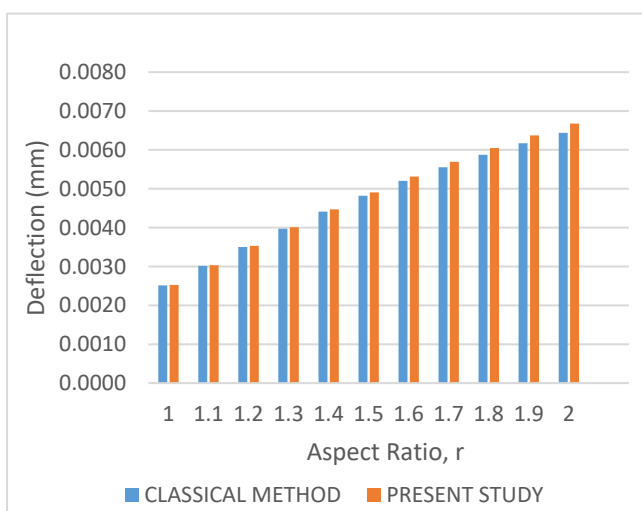
### Appendix A

#### Effect of Aspect Ratio on the Deflection of Membrane and Stiffened Plates and the Percentage Difference Between the Deflections of Kirchhoff's Hypothesis and the Present Study

##### Appendix A1: Membrane plates



##### Appendix A2: Stiffened plates

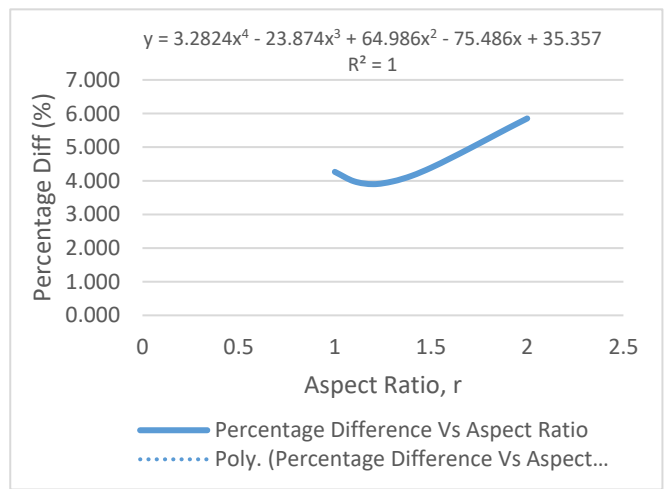
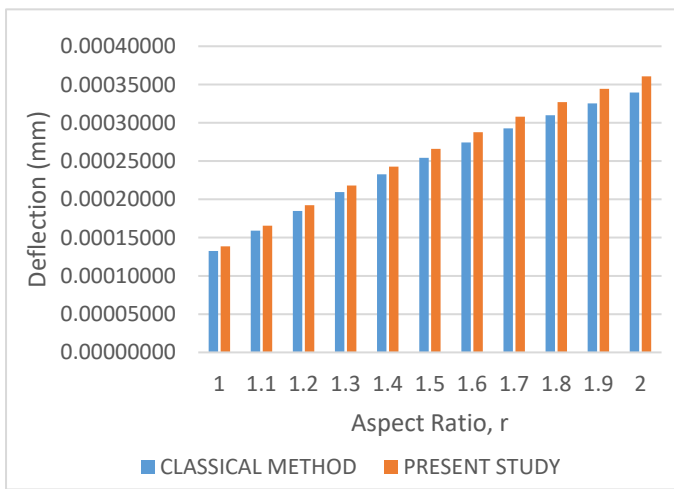




## Appendix B

### Effect of Aspect Ratio on the Deflection of Moderately Thick and Thick Plates and the Percentage Difference Between the Deflections of Kirchhoff's Hypothesis and the Present Study

#### Appendices B1: Moderately Thick Plate



#### Appendices B2: Thick Plate

